Problem 1.31

Solve the following differential equations:

(n) $xy' = y + xe^{y/x}$;

Solution

Divide both sides of the equation by x.

$$y' = \frac{y}{x} + e^{y/x}$$

The right side prompts the substitution,

$$u = \frac{y}{x} \quad \rightarrow \quad xu = y$$
$$u + x\frac{du}{dx} = \frac{dy}{dx}.$$

Plugging these expressions into the ODE, we have

$$u + x\frac{du}{dx} = u + e^u.$$

Cancel u from both sides.

$$x\frac{du}{dx} = e^u$$

This equation can be solved by separation of variables.

$$e^{-u}\,du = \frac{dx}{x}$$

Integrate both sides.

$$-e^{-u} = \ln|x| + C$$

Multiply both sides by -1.

$$e^{-u} = -\ln|x| - C$$

Take the logarithm of both sides.

$$-u = \ln(-\ln|x| - C)$$

Use a new arbitrary constant $\ln B$, remove the minus sign in front of the logarithm by inverting its argument, and multiply both sides by -1 to solve for u.

$$u(x) = -\ln\left(\ln\frac{1}{|x|} + \ln B\right)$$

Now that the integration is done, change back to the original variable y. Combine the logarithms and remove the minus sign in front of the logarithm by inverting its argument.

$$\frac{y}{x} = \ln\frac{1}{\ln\frac{B}{|x|}}$$

The point of using $\ln B$ for the new arbitrary constant is so that B is on top of the absolute value sign here. This allows us to drop the absolute value sign because it doesn't matter whether x is positive or negative. Multiply both sides by x to solve for y. Therefore,

$$y(x) = x\ln\frac{1}{\ln\frac{B}{x}}.$$