Problem 1.31

Solve the following differential equations:

(r) $x^2y'' + xy' - y = 3x^2$ $[y(1) = y(2) = 1]$;

Solution

This is an inhomogeneous ODE, so the general solution is the sum of the complementary solution y_c and the particular solution y_p.

$$y(x) = y_c + y_p$$

We'll start by finding y_c, which is the solution to the associated homogeneous equation.

$$x^2y_c'' + xy_c' - y_c = 0$$

This ODE is equidimensional since the change in scale $x \to ax$ leaves the equation unchanged. Thus, the solution is of the form $y_c = x^r$. Our task now is to plug this expression into the ODE to determine the values of r for which it holds.

$$y_c = x^r \quad \to \quad y_c' = rx^{r-1} \quad \to \quad y_c'' = r(r-1)x^{r-2}$$

Substituting these expressions into the ODE yields

$$r(r-1)x^r + rx^r - x^r = 0.$$

Divide both sides by x^r to obtain the indicial equation.

$$r(r-1) + r - 1 = 0$$

r cancels out.

$$r^2 - 1 = 0$$

Factor the left side.

$$(r-1)(r+1) = 0$$

Thus, $r = 1$ or $r = -1$. We can now write the solution for the associated homogeneous equation.

$$y_c(x) = C_1x^1 + C_2x^{-1}$$

Our next goal is to determine the particular solution y_p. To do this, we will use the method of variation of parameters. That is, we will assume y_p has the form

$$y_p = u_1(x)x + u_2(x)x^{-1},$$

where u_1 and u_2 satisfy

$$xu_1' + x^{-1}u_2' = 0$$

$$u_1' + (-1)x^{-2}u_2' = \frac{3x^2}{x^2} = 3.$$

Solve this system of equations for u_1' and u_2' using Cramer's rule.

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1} \\ 3 & -x^{-2} \end{vmatrix}}{\begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}} = \frac{-\frac{3}{x}}{-\frac{2}{x}} = \frac{3}{2}$$

$$u_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}} = \frac{3x}{-\frac{2}{x}} = -\frac{3}{2}x^2$$

Now that we know u_1' and u_2', we can determine u_1 and u_2 by integration. We're not concerned with the integration constants.

$$u_1(x) = \frac{3}{2}x$$
$$u_2(x) = -\frac{1}{2}x^3$$

Hence, the particular solution is

$$y_p = \frac{3}{2}x^2 - \frac{1}{2}x^2 = x^2.$$

Therefore, the general solution is

$$y(x) = C_1 x + C_2 x^{-1} + x^2.$$

We can now determine the two arbitrary constants, C_1 and C_2, by applying the provided boundary conditions, $y(1) = 1$ and $y(2) = 1$. The result is the following system of equations.

$$y(1) = C_1 + C_2 + 1 = 1$$
$$y(2) = 2C_1 + \frac{C_2}{2} + 4 = 1$$

Solving the system gives us $C_1 = -2$ and $C_2 = 2$. Therefore,

$$y(x) = -2x + \frac{2}{x} + x^2.$$

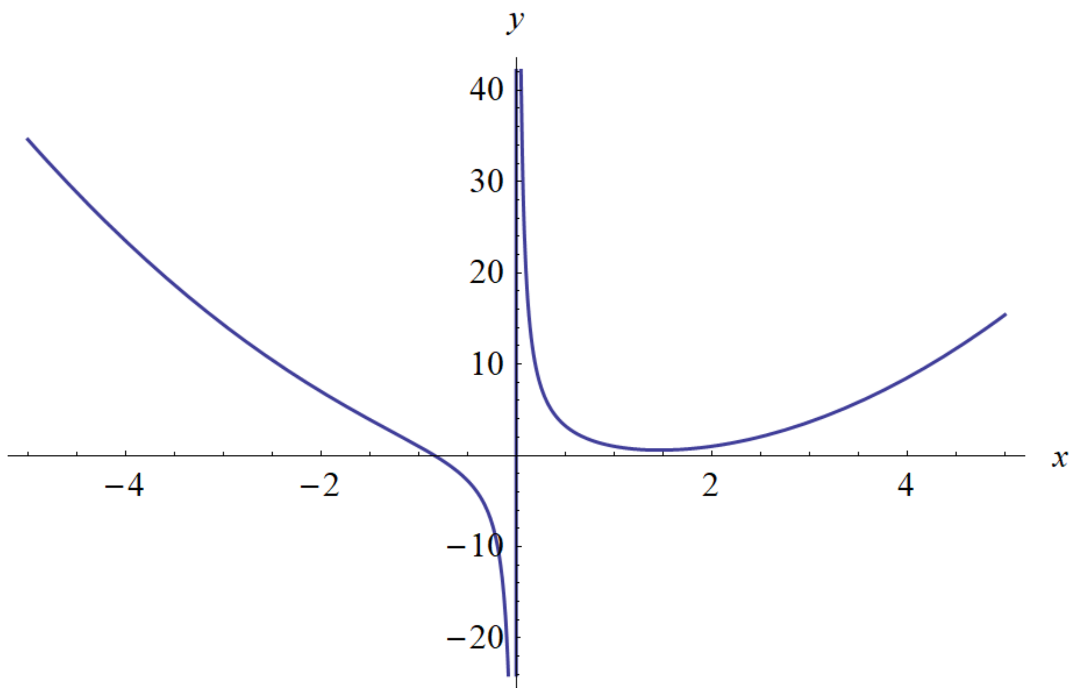

Figure 1: Plot of the solution for $-5 < x < 5$.