Problem 1.31

Solve the following differential equations:

(s) $y^3(y')^2y'' = -\frac{1}{2}$ $[y(0) = y'(0) = 1]$

Solution

This ODE is second-order and autonomous, meaning the independent variable x does not appear in the equation. We can hence make the substitution,

$$y'(x) = u(y)$$
$$y''(x) = \frac{du}{dy}\frac{dy}{dx} = u'(y)u(y),$$

to reduce the equation's order and make it easier to solve. Plugging these expressions into the ODE gives us

$$y^3u^2u'u = -\frac{1}{2},$$

which can be solved by separation of variables.

$$y^3u^3\frac{du}{dy} = -\frac{1}{2}$$

Separate variables.

$$u^3\,du = -\frac{1}{2}y^{-3}\,dy$$

Integrate both sides.

$$\frac{1}{4}u^4 = \frac{1}{4}y^{-2} + \frac{C}{4}$$

Multiply both sides by 4.

$$u^4 = \frac{1}{y^2} + C$$

Take the fourth root of both sides to solve for u.

$$u(y) = \sqrt[4]{\frac{1}{y^2} + C}$$

Now that we have u, change back to the original variable y.

$$y'(x) = \sqrt[4]{\frac{1}{y^2} + C}$$

At this point, use the provided boundary conditions, $y(0) = 1$ and $y'(0) = 1$, to determine the integration constant C.

$$y'(0) = \sqrt[4]{\frac{1}{[y(0)]^2} + C} \quad \to \quad 1 = \sqrt[4]{1 + C} \quad \to \quad C = 0$$

The ODE has thus been simplified to

$$\frac{dy}{dx} = \sqrt[4]{\frac{1}{y^2}} = \frac{1}{y^{1/2}},$$

which can be solved by separation of variables.

$$y^{1/2}\,dy = dx$$

Integrate both sides.

$$\frac{2}{3}y^{3/2} = x + B$$

Use the boundary condition $y(0) = 1$ to determine B.

$$\frac{2}{3} = B$$

So we have

$$\frac{2}{3}y^{3/2} = x + \frac{2}{3}.$$

Multiply both sides by 3/2.

$$y^{3/2} = \frac{3}{2}x + 1$$

Raise both sides to the 2/3 power to solve for y. Therefore,

$$y(x) = \left(\frac{3}{2}x + 1\right)^{2/3}.$$

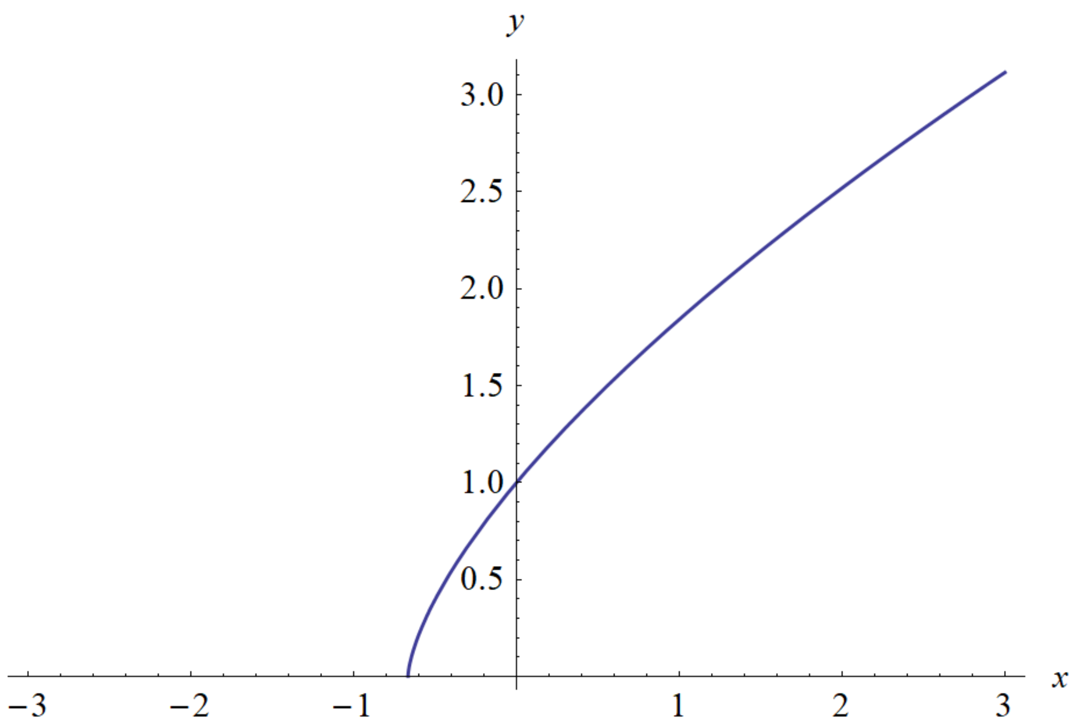

Figure 1: Plot of the solution for $-3 < x < 3$.