Problem 1.31

Solve the following differential equations:

(x) $(x-1)(x-2)y' + y = 2$ $[y(0) = 1]$;

Solution

Divide both sides $(x-1)(x-2)$ to isolate the y' term.

$$y' + \frac{1}{(x-1)(x-2)}y = \frac{2}{(x-1)(x-2)}$$

This is a first-order ODE that can be solved with an integrating factor I.

$$I = e^{\int^x \frac{1}{(s-1)(s-2)}\,ds}$$

To evaluate the integral, use partial fraction decomposition.

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

Our task here is to determine A and B. Multiply both sides by the least common denominator.

$$1 = A(s-2) + B(s-1)$$

Choose two random values of s to get two equations that we can use to solve for A and B.

$$\begin{aligned} s = 2: &\quad 1 = B(1) \\ s = 1: &\quad 1 = A(-1) \end{aligned}$$

The system yields $A = -1$ and $B = 1$, so the integral we have to evaluate in the exponent becomes

$$\int^x \left(-\frac{1}{s-1} + \frac{1}{s-2}\right) ds = -\ln(x-1) + \ln(x-2) = \ln\frac{x-2}{x-1}.$$

Hence,

$$I = e^{\ln\frac{x-2}{x-1}} = \frac{x-2}{x-1}.$$

Multiply both sides of the ODE by this integrating factor.

$$\frac{x-2}{x-1}y' + \frac{1}{(x-1)^2}y = \frac{2}{(x-1)^2}$$

The left side is now exact and can written as $d/dx(Iy)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{x-2}{x-1}y\right) = \frac{2}{(x-1)^2}$$

Integrate both sides of the equation with respect to x.

$$\frac{x-2}{x-1}y = -\frac{2}{x-1} + C$$

Multiply both sides by $x-1$ and divide both sides by $x-2$ to solve for y.

$$y(x) = -\frac{2}{x-2} + \frac{C(x-1)}{x-2}$$

Combine the two terms into one.

$$y(x) = \frac{C(x-1)-2}{x-2}$$

Use the provided initial condition, $y(0) = 1$, to determine C.

$$1 = \frac{C(-1)-2}{-2} \quad \rightarrow \quad C = 0$$

Therefore,

$$y(x) = \frac{2}{2-x}.$$

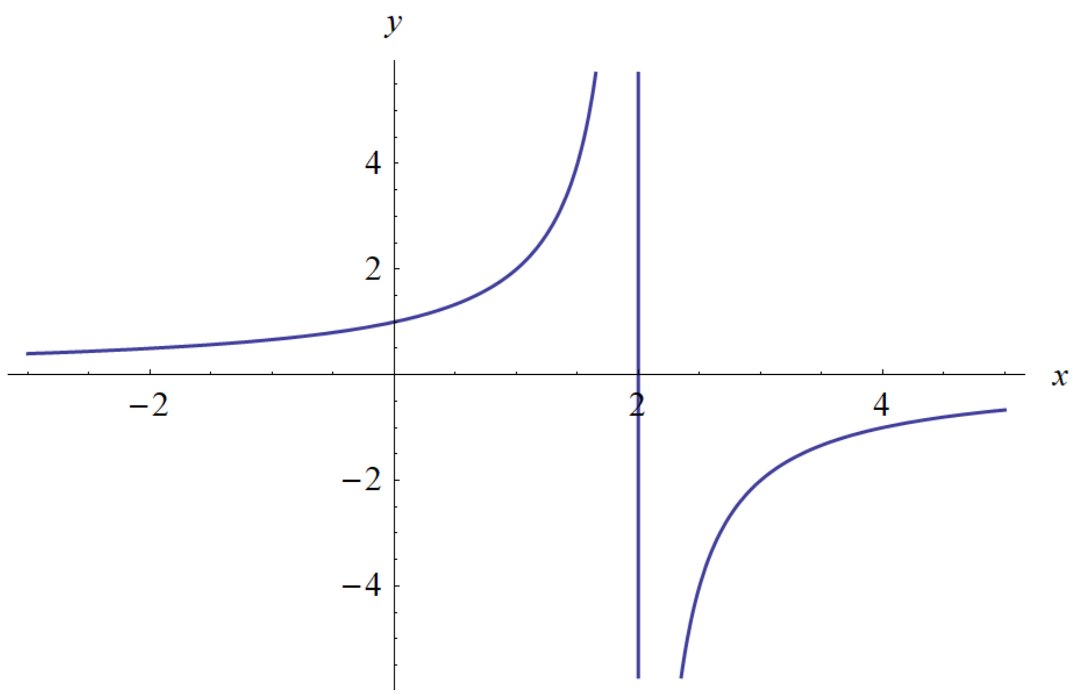

Figure 1: Plot of the solution for $-3 < x < 5$.