Problem 1.31

Solve the following differential equations:

(y) $y' = 1/(x + e^y)$;

Solution

This ODE for $y(x)$ is quite difficult, so invert both sides of the equation.

$$\frac{dy}{dx} = \frac{1}{x + e^y} \quad \to \quad \frac{dx}{dy} = x + e^y$$

Bring x over to the left side.

$$\frac{dx}{dy} - x = e^y$$

This is a simpler first-order inhomogeneous ODE for x that can be solved with an integrating factor I. x is now the dependent variable, and y is now the independent variable.

$$I = e^{\int^y -1\, ds} = e^{-y}$$

Multiply both sides of the equation by I.

$$e^{-y}\frac{dx}{dy} - e^{-y}x = 1$$

The left side is now exact and can be written as $d/dy(Ix)$.

$$\frac{d}{dy}(e^{-y}x) = 1$$

Integrate both sides with respect to y.

$$e^{-y}x = y + C$$

Multiply both sides by e^y to solve for x.

$$x(y) = e^y(y + C)$$

This is an implicit solution for y.