Problem 1.34

Express the solution of the initial-value problem

$$x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}-2\right)\left(x\frac{d}{dx}-3\right)y(x)=f(x), \quad y(1)=y'(1)=y''(1)=y'''(1)=0,$$

as an integral.

Solution

Because the differential operator acting on y is factored, we can solve the ODE by constructing the corresponding system of first-order ODEs. Let

$$\left(x\frac{d}{dx}-3\right)y=y_1. \tag{1}$$

Then the ODE simplifies to

$$x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)\left(x\frac{d}{dx}-2\right)y_1=f(x).$$

Let

$$\left(x\frac{d}{dx}-2\right)y_1=y_2 \tag{2}$$

so that the ODE simplifies to

$$x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)y_2=f(x).$$

Let

$$\left(x\frac{d}{dx}-1\right)y_2=y_3 \tag{3}$$

so that the ODE simplifies to

$$x\frac{d}{dx}y_3=f(x).$$

Our aim now is to solve for y_3, y_2, and y_1 successively and end up with a first-order equation for y that we can solve relatively easily. The equation for y_3 is straightforward to solve.

$$\frac{d}{dx}y_3=\frac{f(x)}{x}$$

Integrate both sides with respect to x.

$$y_3(x)=\int^x\frac{f(r)}{r}\,dr+C_1$$

Now that we know y_3, we can solve for y_2. Plug the solution in to the right side of equation (3) and expand the operator on the left side.

$$x\frac{d}{dx}y_2-y_2=\int^x\frac{f(r)}{r}\,dr+C_1$$

Divide both sides by x.

$$\frac{d}{dx}y_2-\frac{1}{x}y_2=\frac{1}{x}\int^x\frac{f(r)}{r}\,dr+\frac{C_1}{x}$$

This is a first-order inhomogeneous ODE that can be solved with an integrating factor.

$$I = e^{\int^x -\frac{1}{s}\,ds} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

Multiply both sides of the equation by this integrating factor.

$$\frac{1}{x}\frac{d}{dx}y_2 - \frac{1}{x^2}y_2 = \frac{1}{x^2}\int^x \frac{f(r)}{r}\,dr + \frac{C_1}{x^2}$$

The left side is exact and can be written as $d/dx(Iy_2)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{1}{x}y_2\right) = \frac{1}{x^2}\int^x \frac{f(r)}{r}\,dr + \frac{C_1}{x^2}$$

Integrate both sides with respect to x.

$$\frac{1}{x}y_2 = \int^x \frac{1}{s^2}\int^s \frac{f(r)}{r}\,dr\,ds - \frac{C_1}{x} + C_2$$

Multiply both sides by x to solve for y_2.

$$y_2(x) = x\int^x \frac{1}{s^2}\int^s \frac{f(r)}{r}\,dr\,ds - C_1 + C_2x$$

Now that we know y_2, we can solve for y_1. Plug the solution in to the right side of equation (2) and expand the operator on the left side.

$$x\frac{d}{dx}y_1 - 2y_1 = x\int^x \frac{1}{s^2}\int^s \frac{f(r)}{r}\,dr\,ds - C_1 + C_2x$$

Divide both sides by x.

$$\frac{d}{dx}y_1 - \frac{2}{x}y_1 = \int^x \frac{1}{s^2}\int^s \frac{f(r)}{r}\,dr\,ds - \frac{C_1}{x} + C_2$$

This is a first-order inhomogeneous ODE that can be solved with an integrating factor.

$$I = e^{\int^x -\frac{2}{s}\,ds} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

Multiply both sides of the equation by this integrating factor.

$$\frac{1}{x^2}\frac{d}{dx}y_1 - \frac{2}{x^3}y_1 = \frac{1}{x^2}\int^x \frac{1}{s^2}\int^s \frac{f(r)}{r}\,dr\,ds - \frac{C_1}{x^3} + \frac{C_2}{x^2}$$

The left side is now exact and can be written as $d/dx(Iy_1)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{1}{x^2}y_1\right) = \frac{1}{x^2}\int^x \frac{1}{s^2}\int^s \frac{f(r)}{r}\,dr\,ds - \frac{C_1}{x^3} + \frac{C_2}{x^2}$$

Integrate both sides with respect to x.

$$\frac{1}{x^2}y_1 = \int^x \frac{1}{t^2}\int^t \frac{1}{s^2}\int^s \frac{f(r)}{r}\,dr\,ds\,dt + \frac{C_1}{2x^2} - \frac{C_2}{x} + C_3$$

Multiply both sides by x^2 to solve for y_1.

$$y_1(x) = x^2 \int^x \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r}\, dr\, ds\, dt + \frac{C_1}{2} - xC_2 + C_3x^2$$

Now that we know y_1, we can solve for y. Plug the solution in to the right side of equation (1) and expand the operator on the left side.

$$x\frac{d}{dx}y - 3y = x^2 \int^x \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r}\, dr\, ds\, dt + \frac{C_1}{2} - xC_2 + C_3x^2$$

Divide both sides by x.

$$\frac{d}{dx}y - \frac{3}{x}y = x \int^x \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r}\, dr\, ds\, dt + \frac{C_1}{2x} - C_2 + C_3x$$

This is a first-order inhomogeneous ODE that can be solved with an integrating factor.

$$I = e^{\int^x -\frac{3}{s}\, ds} = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$$

Multiply both sides of the equation by this integrating factor.

$$\frac{1}{x^3}\frac{d}{dx}y - \frac{3}{x^4}y = \frac{1}{x^2} \int^x \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r}\, dr\, ds\, dt + \frac{C_1}{2x^4} - \frac{C_2}{x^3} + \frac{C_3}{x^2}$$

The left side is now exact and can be written as $d/dx(Iy_1)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{1}{x^3}y\right) = \frac{1}{x^2} \int^x \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r}\, dr\, ds\, dt + \frac{C_1}{2x^4} - \frac{C_2}{x^3} + \frac{C_3}{x^2}$$

Integrate both sides with respect to x.

$$\frac{1}{x^3}y = \int^x \frac{1}{u^2} \int^u \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r}\, dr\, ds\, dt\, du - \frac{C_1}{6x^3} + \frac{C_2}{2x^2} - \frac{C_3}{x} + C_4$$

Multiply both sides of the equation by x^3 to solve for y.

$$y(x) = x^3 \int^x \frac{1}{u^2} \int^u \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r}\, dr\, ds\, dt\, du - \frac{C_1}{6} + \frac{C_2x}{2} - C_3x^2 + C_4x^3$$

Introduce new constants of integration, A, B, C, and D, to simplify the right side.

$$y(x) = x^3 \int^x \frac{1}{u^2} \int^u \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r}\, dr\, ds\, dt\, du + A + Bx + Cx^2 + Dx^3$$

Now that we have found the general solution for $y(x)$, we have to use the four initial conditions to determine the four constants of integration. We'll start with the first one, $y(1) = 0$. Set the lower limit of the u-integral to be 1 so that the integral term vanishes when $x = 1$ is plugged in.

$$y(1) = A + B + C + D = 0 \tag{4}$$

With the first initial condition applied, $y(x)$ becomes

$$y(x) = x^3 \int_1^x \frac{1}{u^2} \int^u \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r}\, dr\, ds\, dt\, du + A + Bx + Cx^2 + Dx^3.$$

Take the first derivative of y to use the second initial condition, $y'(1) = 0$.

$$y'(x) = 3x^2 \int_1^x \frac{1}{u^2} \int^u \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt \, du + x^3 \frac{1}{x^2} \int^x \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt + B + 2Cx + 3Dx^2$$

Set the lower limit of the t-integral to be 1 so that the second integral term vanishes when $x = 1$ is plugged in.

$$y'(1) = B + 2C + 3D = 0 \tag{5}$$

With the second initial condition applied, $y'(x)$ becomes

$$y'(x) = 3x^2 \int_1^x \frac{1}{u^2} \int_1^u \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt \, du + x \int_1^x \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt + B + 2Cx + 3Dx^2.$$

Take the second derivative of y to use the third initial condition, $y''(1) = 0$.

$$y''(x) = 6x \int_1^x \frac{1}{u^2} \int_1^u \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt \, du + 3x^2 \frac{1}{x^2} \int_1^x \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt + \int_1^x \frac{1}{t^2} \int^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt + x \frac{1}{x^2} \int^x \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds + 2C + 6Dx$$

Set the lower limit of the s-integral to be 1 so that the fourth integral term vanishes when $x = 1$ is plugged in.

$$y''(1) = 2C + 6D = 0 \tag{6}$$

With the third initial condition applied, $y''(x)$ becomes

$$y''(x) = 6x \int_1^x \frac{1}{u^2} \int_1^u \frac{1}{t^2} \int_1^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt \, du + 4 \int_1^x \frac{1}{t^2} \int_1^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt + \frac{1}{x} \int_1^x \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds + 2C + 6Dx.$$

Take the third derivative of y to use the fourth initial condition, $y'''(1) = 0$.

$$y'''(x) = 6 \int_1^x \frac{1}{u^2} \int_1^u \frac{1}{t^2} \int_1^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt \, du + 6x \frac{1}{x^2} \int_1^x \frac{1}{t^2} \int_1^t \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds \, dt + 4 \frac{1}{x^2} \int_1^x \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds - \frac{1}{x^2} \int_1^x \frac{1}{s^2} \int^s \frac{f(r)}{r} \, dr \, ds + \frac{1}{x} \frac{1}{x^2} \int^x \frac{f(r)}{r} \, dr + 6D$$

Set the lower limit of the r-integral to be 1 so that the fifth integral term vanishes when $x = 1$ is plugged in.

$$y'''(1) = 6D = 0 \tag{7}$$

Solving equations (4), (5), (6), and (7) for the constants, we find that $A = 0$, $B = 0$, $C = 0$, and $D = 0$. Therefore,

$$y(x) = x^3 \int_1^x \frac{1}{u^2} \int_1^u \frac{1}{t^2} \int_1^t \frac{1}{s^2} \int_1^s \frac{f(r)}{r} \, dr \, ds \, dt \, du.$$