Problem 3.11

$$\text{If} \quad u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots ,$$
$$v = \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots ,$$
$$w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots ,$$

prove that

$$u^3 + v^3 + w^3 - 3uvw = 1$$

(Putnam Exam 1939).

Solution

The result will be proven by solving a differential equation, which is the easiest way by far. Before beginning, it is important to notice the following relations from the given series.

$$\frac{du}{dx} = w \tag{1}$$
$$\frac{dv}{dx} = u \tag{2}$$
$$\frac{dw}{dx} = v \tag{3}$$

We could solve this system of equations for u, v, and w, but it's unnecessary. Rather, differentiate the left-hand expression implicitly with respect to x and then substitute (1), (2), and (3).

$$\begin{aligned}\frac{d}{dx}(u^3 + v^3 + w^3 - 3uvw) &= 3u^2 \cdot \frac{du}{dx} + 3v^2 \cdot \frac{dv}{dx} + 3w^2 \cdot \frac{dw}{dx} - 3\left(\frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx}\right)\\ &= 3u^2 \cdot w + 3v^2 \cdot u + 3w^2 \cdot v - 3(w^2v + u^2w + v^2u)\\ &= 3u^2w + 3v^2u + 3w^2v - 3w^2v - 3u^2w - 3v^2u\\ &= 0\end{aligned}$$

The differential equation to solve is therefore

$$\frac{d}{dx}(u^3 + v^3 + w^3 - 3uvw) = 0.$$

Integrating both sides, we get

$$u^3 + v^3 + w^3 - 3uvw = C,$$

where C is an arbitrary constant. We can determine this constant conveniently by setting $x = 0$. In this case, $u = 1$, $v = 0$, and $w = 0$. Thus,

$$1 = C,$$

and the desired result is obtained.

$$u^3 + v^3 + w^3 - 3uvw = 1$$