Problem 3.47

Show that

$$x + \frac{2}{3}x^3 + \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)x^5 + \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)x^7 + \cdots = \frac{\text{arc sin}\, x}{\sqrt{1-x^2}}$$

(Putnam Exam 1948).

Clue: What differential equation does this Taylor series satisfy?

Solution

We will start from the right side and derive the series on the left. Following the clue, a differential equation will be written for it, and a series solution will be obtained. Let $y(x)$ represent the unknown series.

$$y(x) = \frac{\text{arc sin}\, x}{\sqrt{1-x^2}} \tag{1}$$

Notice that one function is the derivative of the other on the right side.

$$\frac{d}{dx}(\text{arc sin}\, x) = \frac{1}{\sqrt{1-x^2}}$$

So we have

$$y(x) = \text{arc sin}\, x \frac{d}{dx}(\text{arc sin}\, x).$$

The right side can be written in the following way.

$$y(x) = \frac{d}{dx}\left[\frac{(\text{arc sin}\, x)^2}{2}\right]$$

Taking the derivative of the term in square brackets and using the chain rule gives us the equation in the previous step. The point of writing it this way is that now the entire right side is the derivative of some function. We can get rid of the derivative by integrating both sides with respect to x.

$$\int y(x)\, dx = \frac{(\text{arc sin}\, x)^2}{2}$$

Multiply both sides by 2 to eliminate the fraction.

$$2\int y(x)\, dx = (\text{arc sin}\, x)^2 \tag{2}$$

This is not the only way to get $(\text{arc sin}\, x)^2$. We could also multiply both sides of equation (1) by $\sqrt{1-x^2}$ and square both sides to get it on the right side.

$$(1-x^2)[y(x)]^2 = (\text{arc sin}\, x)^2 \tag{3}$$

Combining equations (2) and (3) gives us an integral equation for $y(x)$.

$$(1-x^2)[y(x)]^2 = 2\int y(x)\, dx$$

Take the derivative of both sides with respect to x to turn this into a differential equation.

$$\frac{d}{dx}\{(1-x^2)[y(x)]^2\} = 2y(x)$$

Evaluate the derivative on the left side.

$$-2x[y(x)]^2 + (1 - x^2) \cdot 2y(x)y'(x) = 2y(x)$$

Divide both sides by $2y(x)$. From now on, the dependence of y on x will only be implicit.

$$-xy + (1 - x^2)y' = 1$$

This is the differential equation we sought to obtain from the beginning. It is first order and inhomogeneous, but we will not be solving it with an integrating factor. Instead, we need to find the series solution, so assume a power series representation for y. This is appropriate because $x = 0$ is an ordinary point.

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

We can determine the coefficients a_n by substituting these expressions for y and y' into the differential equation. Before we do so, write the differential equation like so.

$$y' - x^2 y' - xy = 1$$

Now plug in the expressions.

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - x^2 \sum_{n=0}^{\infty} n a_n x^{n-1} - x \sum_{n=0}^{\infty} a_n x^n = 1$$

Bring the variables into the summands.

$$\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 1$$

In order to combine the sums they all have to start from the same n value and the exponents of x have to be the same in each summand. The only sum to modify is the first one on the left. Write out the first two terms of it.

$$0 + a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 1$$

We can start the first sum from $n = 0$ as long as we replace n with $n + 2$ everywhere in the summand.

$$a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} x^{n+1} - \sum_{n=0}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 1$$

Now the sums can be combined.

$$a_1 + \sum_{n=0}^{\infty} [(n+2) a_{n+2} x^{n+1} - n a_n x^{n+1} - a_n x^{n+1}] = 1$$

Factor the summand.

$$a_1 + \sum_{n=0}^{\infty} [(n+2) a_{n+2} - (n+1) a_n] x^{n+1} = 1$$

Now we match coefficients on both sides of the equation. Since there are no terms with x on the right side, the coefficients of x^{n+1} must be equal to 0.

$$a_1 = 1 \qquad\qquad (n+2)a_{n+2} - (n+1)a_n = 0$$

Solving the recurrence relation for a_{n+2} gives us

$$a_{n+2} = \frac{n+1}{n+2}a_n,$$

so the odd coefficients are

$$\begin{aligned} a_3 &= \frac{2}{3}a_1 = \frac{2}{3} \\ a_5 &= \frac{4}{5}a_3 = \frac{2}{3} \cdot \frac{4}{5} \\ a_7 &= \frac{6}{7}a_5 = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \\ &\vdots \end{aligned}$$

a_0, the coefficient of x^0, is the one arbitrary constant because the differential equation we solved was first order. We can determine a_0, however, by setting $x = 0$ in equation (1). Doing so wipes out all the terms in the power series except for a_0.

$$y(0) = a_0 = \left.\frac{\arcsin x}{\sqrt{1-x^2}}\right|_{x=0} = 0$$

As a result of the recurrence relation, a_2, a_4, and all other even coefficients are 0. The power series solution for y is thus

$$y(x) = x + \frac{2}{3}x^3 + \frac{2}{3} \cdot \frac{4}{5}x^5 + \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7}x^7 + \cdots . \tag{4}$$

Combining equations (1) and (4), we obtain the desired result,

$$x + \frac{2}{3}x^3 + \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)x^5 + \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)x^7 + \cdots = \frac{\arcsin x}{\sqrt{1-x^2}}.$$