Problem 21

A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.

(a) Write a differential equation for the amount of chemical in the pond at any time.

(b) How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?

Solution

Part (a)

Apply the law of conservation of mass, which states that matter is neither created nor destroyed. If the chemical flows into the pond at some rate, then it must flow out at that same rate or else the chemical will accumulate in the pond (assuming it flows in faster). This idea is expressed mathematically as follows.

$$\text{rate of chemical accumulation} = \text{rate of chemical in} - \text{rate of chemical out}$$

Let $m = m(t)$ denote the chemical mass in grams and let t denote the time in hours. The rates will then be in units of grams per hour.

$$\begin{aligned} \text{rate of chemical accumulation} &= \frac{dm}{dt} \\ \text{rate of chemical in} &= \text{chemical density flowing in} \times \text{volumetric flow rate in} \\ &= 0.01\ \frac{\text{g}}{\cancel{\text{gal}}} \times 300\ \frac{\cancel{\text{gal}}}{\text{h}} = 3\ \frac{\text{g}}{\text{h}} \\ \text{rate of chemical out} &= \text{chemical density flowing out} \times \text{volumetric flow rate out} \\ &= \frac{m}{1\,000\,000}\ \frac{\text{g}}{\cancel{\text{gal}}} \times 300\ \frac{\cancel{\text{gal}}}{\text{h}} = \frac{3m}{10\,000}\ \frac{\text{g}}{\text{h}} \end{aligned}$$

The assumption that the chemical is uniformly distributed in the pond allows us to write the chemical density flowing out as we have. By the law of conservation of mass then,

$$\frac{dm}{dt} = 3 - \frac{3m}{10\,000},$$

and the initial condition associated with it is $m(0) = m_0$, where m_0 is the initial mass in the pond.

Part (b)

After a long time, the chemical mass in the pond will stop changing, so $dm/dt \to 0$ as $t \to \infty$.

$$0 = 3 - \frac{3m}{10\,000}$$

Solving for m, the mass at equilibrium is therefore

$$m = 10\,000 \text{ g},$$

independent of how much is present initially.