Problem 22

A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.

Solution

Considering that evaporation leads to a loss of mass, we have the following proportionality.

$$\underbrace{-\frac{dm}{dt}}_{\text{rate of evaporation}} \quad \underbrace{\propto}_{\text{is proportional to}} \quad \underbrace{A}_{\text{surface area}}$$

The mass itself is density times volume.

$$-\frac{d(\rho V)}{dt} \propto A$$

The density of the raindrop is assumed to be constant.

$$-\rho\frac{dV}{dt} \propto A \tag{1}$$

The formulas for the volume and surface area of a sphere are

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \quad \rightarrow \quad \begin{cases} r = \sqrt[3]{\dfrac{3V}{4\pi}} \\ \dfrac{3V}{r} = 4\pi r^2 \end{cases} \\ A &= 4\pi r^2. \end{aligned}$$

From the second equation in the curly brace, we have

$$\frac{3V}{\sqrt[3]{\frac{3V}{4\pi}}} = A$$

$$6^{2/3}\pi^{1/3}V^{2/3} = A.$$

Substitute this result for A into equation (1).

$$-\rho\frac{dV}{dt} \propto 6^{2/3}\pi^{1/3}V^{2/3}$$

Divide both sides by $-\rho$.

$$\frac{dV}{dt} \propto -\frac{6^{2/3}\pi^{1/3}}{\rho}V^{2/3}$$

In order to change this proportionality into an equation, introduce the positive proportionality constant κ on the right side.

$$\frac{dV}{dt} = -\frac{6^{2/3}\pi^{1/3}}{\rho}\kappa V^{2/3}$$

Therefore, letting k be the positive constant in front of $V^{2/3}$,

$$\frac{dV}{dt} = -kV^{2/3}.$$