Problem 24

A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4 (h)$^{-1}$.

(a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.

(b) How much of the drug is present in the bloodstream after a long time?

Solution

Part (a)

Apply the law of conservation of mass, which states that matter is neither created nor destroyed. If the drug flows into the bloodstream at some rate, then it must flow out at the same rate unless some of it is absorbed. This idea is expressed mathematically as follows.

$$\text{rate of drug absorption} = \text{rate of drug in} - \text{rate of drug out}$$

Let $m = m(t)$ denote the drug mass in milligrams and let t denote the time in hours. The rates will then be in units of milligrams per hour.

$$\begin{aligned}
\text{rate of drug absorption} &= \frac{dm}{dt} \\
\text{rate of chemical in} &= \text{drug density flowing in} \times \text{volumetric flow rate in} \\
&= 5\ \frac{\text{mg}}{\cancel{\text{cm}^3}} \times 100\ \frac{\cancel{\text{cm}^3}}{\text{h}} = 500\ \frac{\text{mg}}{\text{h}} \\
\text{rate of drug out} &\propto m\ \text{mg} \\
&= 0.4m\ \frac{\text{mg}}{\text{h}}
\end{aligned}$$

By the law of conservation of mass then,

$$\frac{dm}{dt} = 500 - 0.4m,$$

and the initial condition associated with it is $m(0) = 0$.

Part (b)

After a long time, the mass of drug in the bloodstream will stop changing, so $dm/dt \to 0$ as $t \to \infty$.

$$0 = 500 - 0.4m$$

Solving for m, the mass at equilibrium is therefore

$$m = 1250\ \text{mg},$$

independent of how much is present initially.