Problem 2

Follow the instructions for Problem 1 for the following initial value problems:

(a) $dy/dt = y - 5, \qquad y(0) = y_0$

(b) $dy/dt = 2y - 5, \qquad y(0) = y_0$

(c) $dy/dt = 2y - 10, \qquad y(0) = y_0$

Solution

Part (a)

$$y' = y - 5$$

Divide both sides by $y - 5$.

$$\frac{y'}{y-5} = 1$$

The left side can be written as $d/dt(\ln|y-5|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}\ln|y-5| = 1$$

Integrate both sides with respect to t.

$$\ln|y-5| = t + C$$

Exponentiate both sides.

$$\begin{aligned} |y-5| &= e^{t+C} \\ &= e^C e^t \end{aligned}$$

Introduce $\pm$ on the right side in order to remove the absolute value sign.

$$y - 5 = \pm e^C e^t$$

Let $A = \pm e^C$ and add 5 to both sides to solve for y.

$$y(t) = 5 + Ae^t$$

Apply the initial condition now to determine A.

$$y(0) = 5 + A = y_0 \quad \rightarrow \quad A = y_0 - 5$$

Therefore,

$$y(t) = 5 + (y_0 - 5)e^t.$$

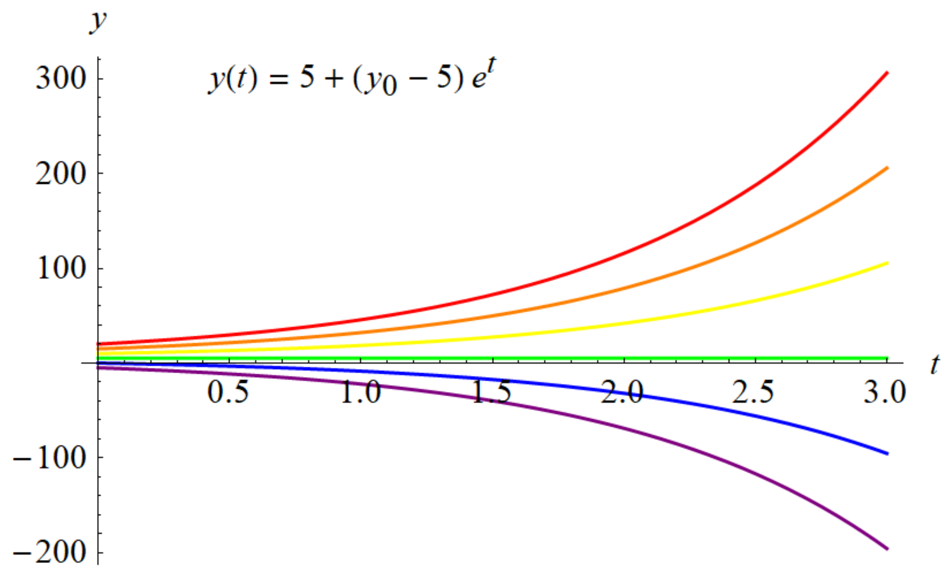

Figure 1: In this figure $y(t)$ is plotted versus t for various values of y_0. The curves in red, orange, yellow, green, blue, and purple correspond to $y_0 = 20$, $y_0 = 15$, $y_0 = 10$, $y_0 = 5$, $y_0 = 0$, and $y_0 = -5$, respectively. The equilibrium solution is at $y = 5$, and the nonequilibrium solutions diverge from it more slowly than in (b) and (c).

Part (b)

$$\begin{aligned} y' &= 2y - 5 \\ &= 2\left(y - \frac{5}{2}\right) \end{aligned}$$

Divide both sides by $y - \frac{5}{2}$.

$$\frac{y'}{y - \frac{5}{2}} = 2$$

The left side can be written as $d/dt(\ln|y - 5/2|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt} \ln\left|y - \frac{5}{2}\right| = 2$$

Integrate both sides with respect to t.

$$\ln\left|y - \frac{5}{2}\right| = 2t + C$$

Exponentiate both sides.

$$\begin{aligned} \left|y - \frac{5}{2}\right| &= e^{2t+C} \\ &= e^C e^{2t} \end{aligned}$$

Introduce $\pm$ on the right side in order to remove the absolute value sign.

$$y - \frac{5}{2} = \pm e^C e^{2t}$$

Let $A = \pm e^C$ and add $5/2$ to both sides to solve for y.

$$y(t) = \frac{5}{2} + Ae^{2t}$$

Apply the initial condition now to determine A.

$$y(0) = \frac{5}{2} + A = y_0 \quad \rightarrow \quad A = y_0 - \frac{5}{2}$$

Therefore,

$$y(t) = \frac{5}{2} + \left(y_0 - \frac{5}{2}\right) e^{2t}.$$

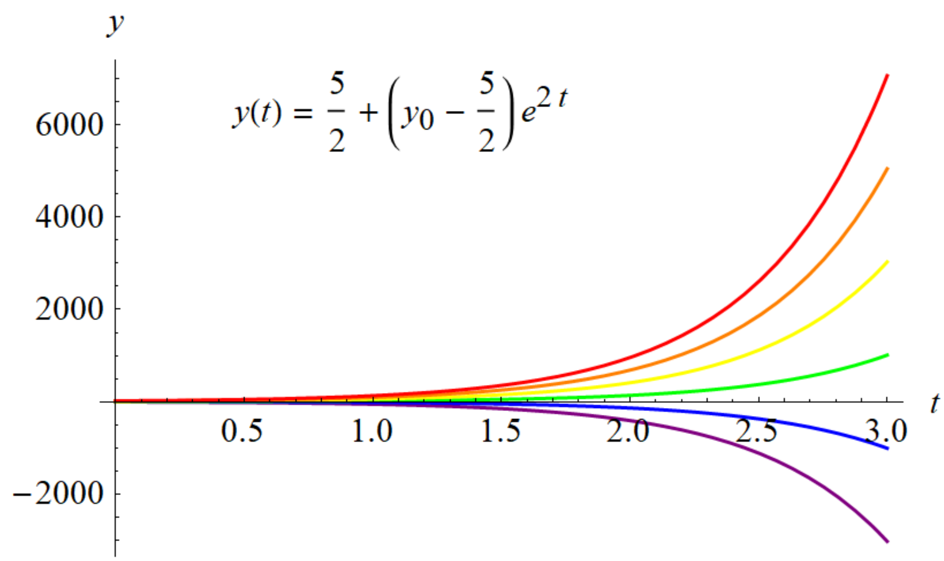

Figure 2: In this figure $y(t)$ is plotted versus t for various values of y_0. The curves in red, orange, yellow, green, blue, and purple correspond to $y_0 = 20$, $y_0 = 15$, $y_0 = 10$, $y_0 = 5$, $y_0 = 0$, and $y_0 = -5$, respectively. The equilibrium solution is at $y = 5/2$, and the nonequilibrium solutions diverge from it faster than in (a).

Part (c)

$$\begin{aligned} y' &= 2y - 10 \\ &= 2(y - 5) \end{aligned}$$

Divide both sides by $y - 5$.

$$\frac{y'}{y - 5} = 2$$

The left side can be written as $d/dt(\ln|y-5|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}\ln|y-5| = 2$$

Integrate both sides with respect to t.

$$\ln|y-5| = 2t + C$$

Exponentiate both sides.

$$\begin{aligned}|y-5| &= e^{2t+C}\\ &= e^C e^{2t}\end{aligned}$$

Introduce $\pm$ on the right side in order to remove the absolute value sign.

$$y - 5 = \pm e^C e^{2t}$$

Let $A = \pm e^C$ and add 5 to both sides to solve for y.

$$y(t) = 5 + Ae^{2t}$$

Apply the initial condition now to determine A.

$$y(0) = 5 + A = y_0 \quad \rightarrow \quad A = y_0 - 5$$

Therefore,

$$y(t) = 5 + (y_0 - 5)e^{2t}.$$

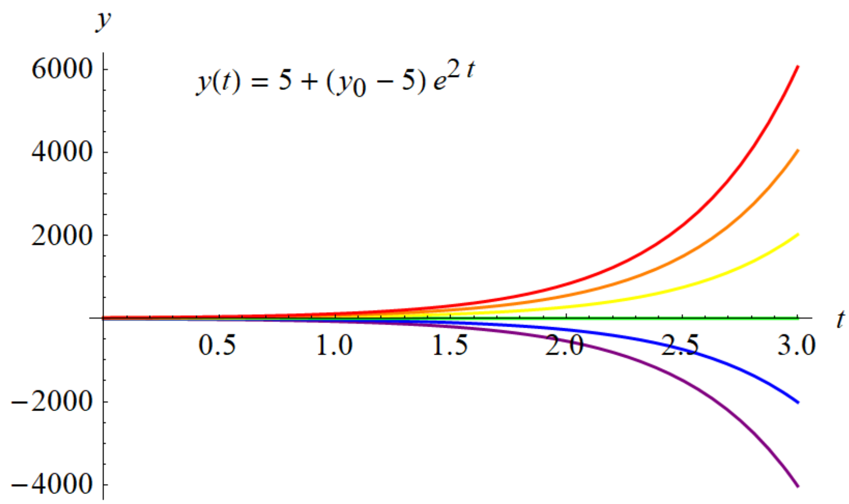

Figure 3: In this figure $y(t)$ is plotted versus t for various values of y_0. The curves in red, orange, yellow, green, blue, and purple correspond to $y_0 = 20$, $y_0 = 15$, $y_0 = 10$, $y_0 = 5$, $y_0 = 0$, and $y_0 = -5$, respectively. The equilibrium solution is at $y = 5$, and the nonequilibrium solutions diverge from it faster than in (a).