Problem 7

The field mouse population in Example 1 satisfies the differential equation

$$dp/dt = 0.5p - 450.$$

(a) Find the time at which the population becomes extinct if $p(0) = 850$.

(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.

(c) Find the initial population p_0 if the population is to become extinct in 1 year.

Solution

Note that t is measured in months in this problem.

$$\begin{aligned} p' &= 0.5p - 450 \\ &= 0.5(p - 900) \end{aligned}$$

Divide both sides by $p - 900$.

$$\frac{p'}{p - 900} = 0.5$$

The left side can be written as $d/dt(\ln|p - 900|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt} \ln|p - 900| = 0.5$$

Integrate both sides with respect to t.

$$\ln|p - 900| = 0.5t + C$$

Exponentiate both sides.

$$\begin{aligned} |p - 900| &= e^{0.5t + C} \\ &= e^C e^{0.5t} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$p(t) - 900 = \pm e^C e^{0.5t}$$

Let $A = \pm e^C$ and add 900 to both sides to solve for $p(t)$.

$$p(t) = 900 + Ae^{0.5t}$$

This is the general solution to the ODE.

Part (a)

Apply the initial condition $p(0) = 850$ to determine A.

$$p(0) = 900 + A = 850 \quad \rightarrow \quad A = -50$$

The mouse population is thus given by

$$p(t) = 900 - 50e^{0.5t}.$$

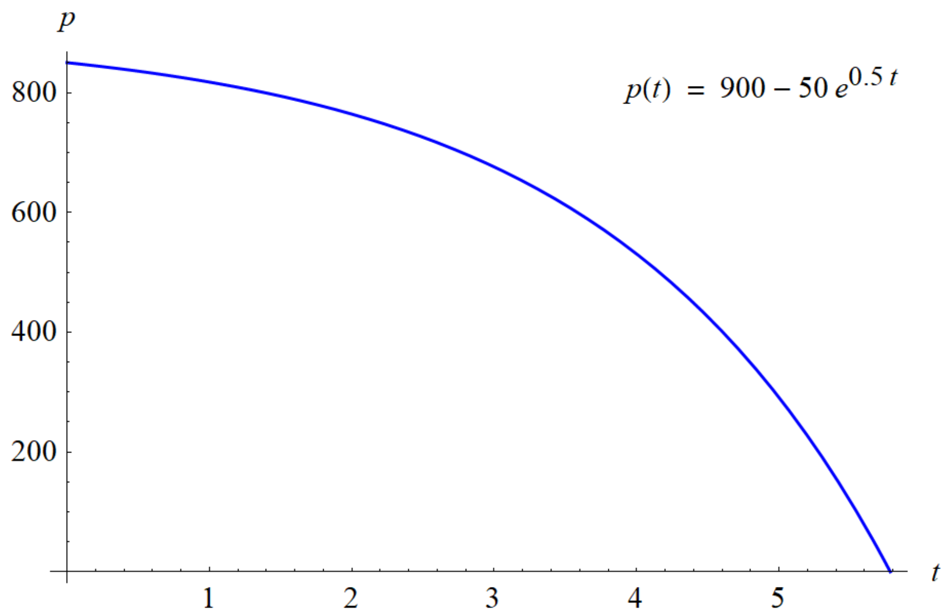

Figure 1: This figure illustrates the mouse population as a function of time if there are 850 mice initially. Time is measured in months.

In order to find the time at which the mice become extinct, set $p = 0$ and solve the resulting equation for t.

$$\begin{aligned} 0 &= 900 - 50e^{0.5t} \\ 50e^{0.5t} &= 900 \\ e^{0.5t} &= 18 \\ \ln e^{0.5t} &= \ln 18 \\ 0.5t \ln e &= \ln 18 \\ 0.5t &= \ln 18 \\ t &= 2 \ln 18 \approx 5.78 \text{ months} \end{aligned}$$

Part (b)

Apply the initial condition $p(0) = p_0$ to determine A.

$$p(0) = 900 + A = p_0 \quad \rightarrow \quad A = p_0 - 900 = -(900 - p_0)$$

The mouse population is thus given by

$$p(t) = 900 - (900 - p_0)e^{0.5t}.$$

In order to find the time at which the mice become extinct, set $p = 0$ and solve the resulting equation for t.

$$\begin{aligned} 0 &= 900 - (900 - p_0)e^{0.5t} \\ (900 - p_0)e^{0.5t} &= 900 \\ e^{0.5t} &= \frac{900}{900 - p_0} \\ \ln e^{0.5t} &= \ln \frac{900}{900 - p_0} \\ 0.5t \ln e &= \ln \frac{900}{900 - p_0} \\ 0.5t &= \ln \frac{900}{900 - p_0} \end{aligned}$$

Therefore,

$$t = 2 \ln \frac{900}{900 - p_0}, \quad 0 < p_0 < 900.$$

Part (c)

If instead we want to find the initial population p_0 given that the population becomes extinct in 1 year, then set $t = 12$ months and $p = 0$ and solve the resulting equation for p_0.

$$\begin{aligned} 0 &= 900 - (900 - p_0)e^{0.5 \times 12} \\ 0 &= 900 - (900 - p_0)e^{6} \\ (900 - p_0)e^{6} &= 900 \\ 900 - p_0 &= \frac{900}{e^6} \\ -p_0 &= -900 + \frac{900}{e^6} \end{aligned}$$

Therefore,

$$p_0 = 900 - \frac{900}{e^6} \approx 897 \text{ mice}.$$