Problem 9

The falling object in Example 2 satisfies the initial value problem

$$dv/dt = 9.8 - (v/5), \qquad v(0) = 0.$$

(a) Find the time that must elapse for the object to reach 98% of its limiting velocity.

(b) How far does the object fall in the time found in part (a)?

Solution

Start by solving the initial value problem. Note that t is measured in seconds, and the velocity is measured in meters/second.

$$\begin{aligned} v' &= 9.8 - \frac{v}{5} \\ &= -\frac{1}{5}(v - 49) \end{aligned}$$

Divide both sides by $v - 49$.

$$\frac{v'}{v - 49} = -\frac{1}{5}$$

The left side can be written as $d/dt(\ln|v - 49|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}\ln|v - 49| = -\frac{1}{5}$$

Integrate both sides with respect to t.

$$\ln|v - 49| = -\frac{1}{5}t + C_1$$

Exponentiate both sides.

$$\begin{aligned} |v - 49| &= e^{-\frac{1}{5}t + C_1} \\ &= e^{C_1}e^{-t/5} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$v(t) - 49 = \pm e^{C_1}e^{-t/5}$$

Let $A = \pm e^{C_1}$ and add 49 to both sides to solve for $v(t)$.

$$v(t) = 49 + Ae^{-t/5}$$

Apply the initial condition $v(0) = 0$ now to determine A.

$$v(0) = 49 + A = 0 \quad \rightarrow \quad A = -49$$

The solution to the initial value problem is

$$v(t) = 49 - 49e^{-t/5}.$$

Therefore,

$$v(t) = 49(1 - e^{-t/5}). \tag{1}$$

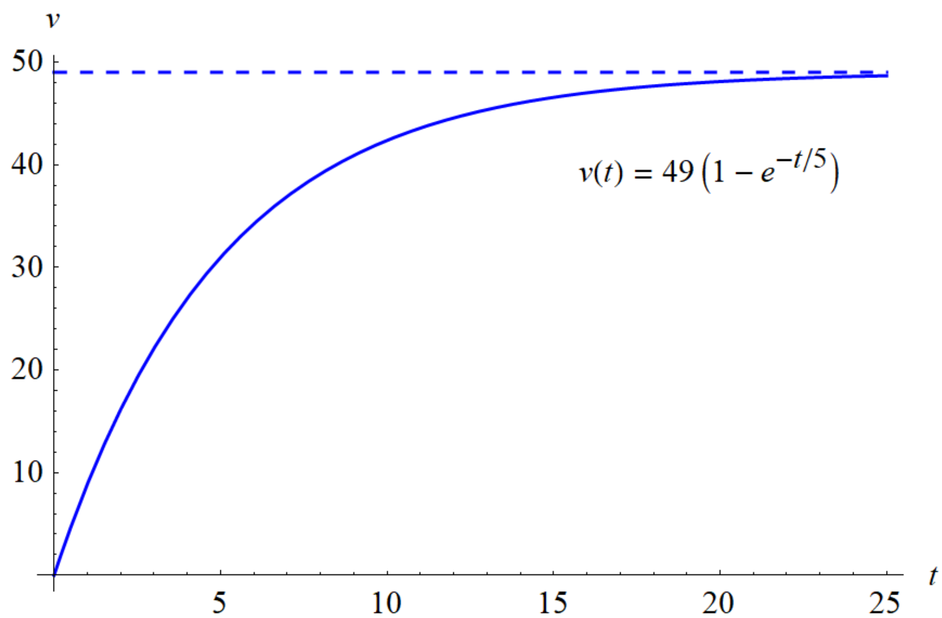

Figure 1: This figure illustrates the velocity of the falling object as a function of time if it starts from rest initially.

Part (a)

From equation (1) we see that the limiting velocity is

$$\lim_{t\to\infty} v(t) = 49.$$

In order to find the time for the object to reach 98% of its limiting velocity, set $v = 0.98(49)$ and solve the resulting equation for t.

$$\begin{aligned}
0.98(49) &= 49(1 - e^{-t/5}) \\
0.98 &= 1 - e^{-t/5} \\
-0.02 &= -e^{-t/5} \\
e^{-t/5} &= 0.02 \\
\ln e^{-t/5} &= \ln 0.02 \\
-\frac{t}{5} \ln e &= \ln 0.02 \\
-\frac{t}{5} &= \ln 0.02
\end{aligned}$$

Therefore,

$$t = -5 \ln 0.02 \approx 19.56 \text{ s}.$$

Part (b)

Velocity is the rate of change of position with respect to time.

$$\frac{dx}{dt} = v(t)$$

To solve for x, integrate both sides with respect to t.

$$\begin{aligned} x(t) &= \int v(t)\, dt + C_2 \\ &= \int 49(1 - e^{-t/5})\, dt + C_2 \\ &= 49\left(\int dt - \int e^{-t/5}\, dt\right) + C_2 \\ &= 49(t + 5e^{-t/5}) + C_2 \end{aligned}$$

Use the initial condition $x(0) = 0$ to determine C_2.

$$x(0) = 49(5) + C_2 = 0 \quad \rightarrow \quad C_2 = -49(5)$$

The position of the falling object is thus

$$\begin{aligned} x(t) &= 49(t + 5e^{-t/5}) - 49(5) \\ &= 49(t + 5e^{-t/5} - 5) \\ &= 49[t + 5(e^{-t/5} - 1)]. \end{aligned}$$

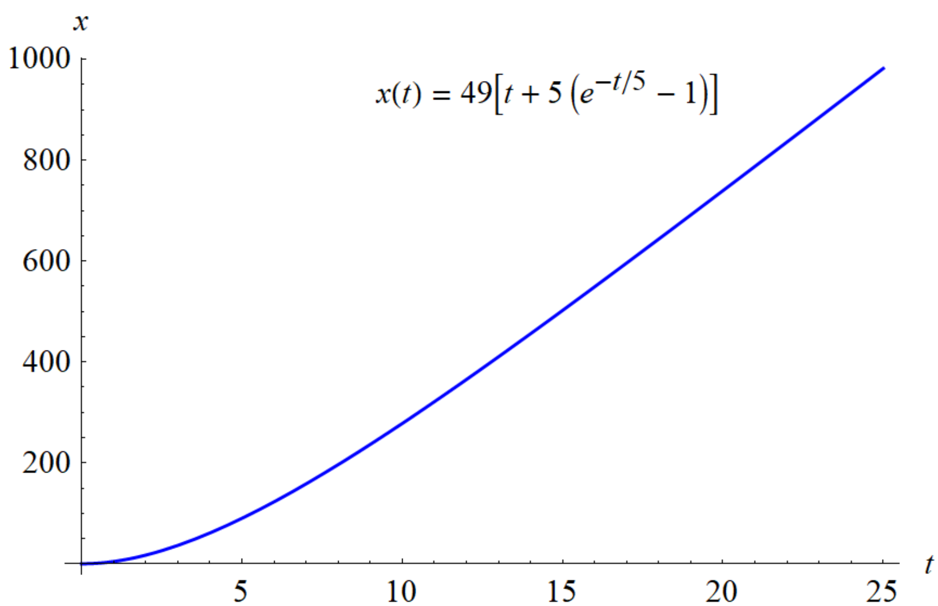

Figure 2: This figure illustrates the position of the falling object as a function of time if it starts from zero initially.

Evaluate $x(-5\ln 0.02)$ to find how far the object falls in the time found in part (a).

$$x(-5\ln 0.02) \approx 718.3 \text{ m}$$