Problem 11

Consider the falling object of mass 10 kg in Example 2, but assume now that the drag force is proportional to the square of the velocity.

(a) If the limiting velocity is 49 m/s, (the same as in Example 2), show that the equation of motion can be written as

$$dv/dt = [(49)^2 - v^2]/245.$$

Also see Problem 25 of Section 1.1.

(b) If $v(0) = 0$, find an expression for $v(t)$ at any time.

(c) Plot your solution from part (b) and the solution (26) from Example 2 on the same axes.

(d) Based on your plots in part (c), compare the effect of a quadratic drag force with that of a linear drag force.

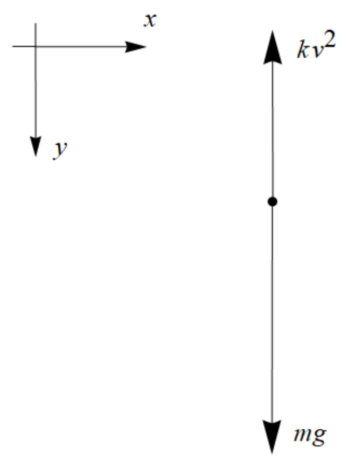
(e) Find the distance $x(t)$ that the object falls in time t.

(f) Find the time T it takes the object to fall 300 m.

Solution

Part (a)

Let k be the proportionality constant so that the drag force is $F_d = kv^2$. There are two forces acting on a falling object, the gravitational force mg and the resistive drag force kv^2, as illustrated in the free-body diagram below.



Newton's second law states that the sum of the forces is equal to mass times acceleration.

$$\sum \mathbf{F} = m\mathbf{a}$$

This vector equation represents the following two scalar equations in the chosen coordinate system.

$$\sum F_x = ma_x$$
$$\sum F_y = ma_y$$

Apply Newton's law to the falling object.

$$0 = 0$$
$$mg - kv^2 = ma$$

Acceleration is the rate of change of velocity with respect to time.

$$mg - kv^2 = m\frac{dv}{dt}$$

Since we're told the limiting velocity is 49 m/s, we can calculate k by setting $dv/dt = 0$ and $v = 49$ and solving the resulting equation for it.

$$mg - k(49^2) = 0 \quad \rightarrow \quad k = \frac{mg}{49^2}$$

As a result, the previous equation becomes

$$mg - \frac{mg}{49^2}v^2 = m\frac{dv}{dt}$$

Divide both sides by m.

$$\begin{aligned}\frac{dv}{dt} &= g - \frac{g}{49^2}v^2 \\ &= g\left(1 - \frac{v^2}{49^2}\right) \\ &= g\frac{49^2 - v^2}{49^2}\end{aligned}$$

Therefore, plugging in 9.8 for g, the governing equation for the velocity is

$$\frac{dv}{dt} = \frac{49^2 - v^2}{245}.$$

Part (b)

The aim here is to solve the initial value problem,

$$v' = \frac{49^2 - v^2}{245}, \quad v(0) = 0.$$

Divide both sides by $49^2 - v^2$.

$$\frac{v'}{49^2 - v^2} = \frac{1}{245}$$

The left side can be written as $d/dt[(1/49)\tanh^{-1}(v/49)]$ by the chain rule.

$$\frac{d}{dt}\left(\frac{1}{49}\tanh^{-1}\frac{v}{49}\right) = \frac{1}{245}$$

Integrate both sides with respect to t.

$$\frac{1}{49}\tanh^{-1}\frac{v}{49} = \frac{1}{245}t + C_1$$

Apply the initial condition now to determine C_1.

$$0 = C_1$$

Consequently, the previous equation becomes

$$\frac{1}{49}\tanh^{-1}\frac{v}{49} = \frac{1}{245}t.$$

Multiply both sides by 49.

$$\tanh^{-1}\frac{v}{49} = \frac{t}{5}$$

Take the hyperbolic tangent of both sides.

$$\frac{v}{49} = \tanh\frac{t}{5}$$

Therefore, the velocity at any time is

$$v(t) = 49\tanh\frac{t}{5}.$$

Part (c)

Equation (26) in the textbook is the solution to the falling body problem with a linear drag force.

$$v = 49(1 - e^{-t/5}) \tag{26}$$

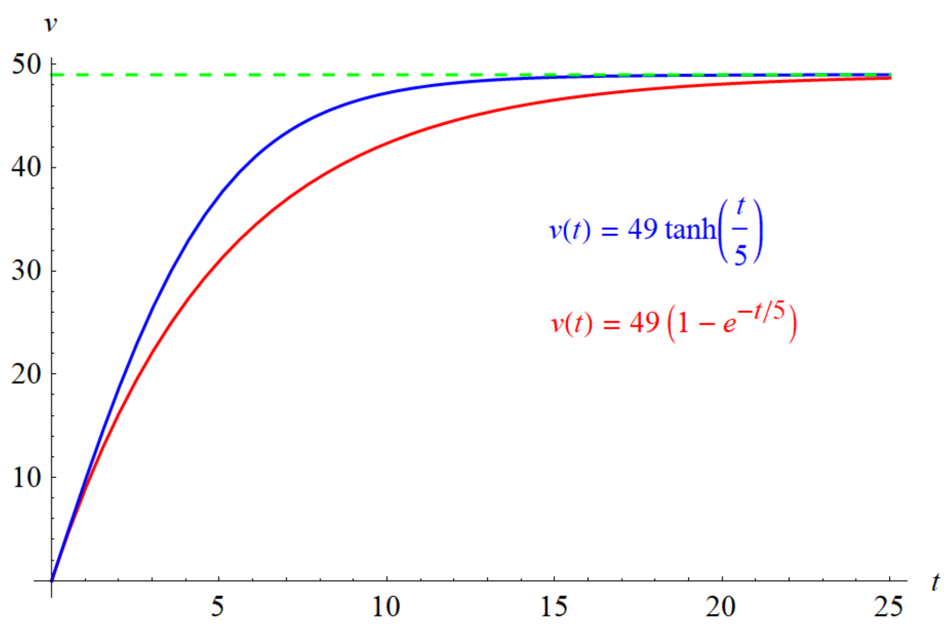


Figure 1: This figure shows a plot of the two functions versus time.

Part (d)

A falling body subject to a quadratic resistive force approaches the terminal velocity faster.

Part (e)

Velocity is the rate of change of position with respect to time.

$$\frac{dx}{dt} = v(t)$$

To solve for x, integrate both sides with respect to t.

$$\begin{aligned} x(t) &= \int v(t)\,dt + C_2 \\ &= \int 49\tanh\frac{t}{5}\,dt + C_2 \\ &= 49 \cdot 5 \ln\cosh\frac{t}{5} + C_2 \end{aligned}$$

Apply the initial condition $x(0) = 0$ to evaluate C_2.

$$x(0) = C_2 = 0$$

Therefore, the position at any time t is

$$x(t) = 245 \ln\cosh\frac{t}{5}.$$

Part (f)

In order to find the time it takes the object to fall 300 meters, set $x = 300$ and $t = T$ and solve the resulting equation for T.

$$\begin{aligned} 300 &= 245\ln\cosh\frac{T}{5} \\ \ln\cosh\frac{T}{5} &= \frac{60}{49} \\ \cosh\frac{T}{5} &= \exp\left(\frac{60}{49}\right) \\ \frac{T}{5} &= \cosh^{-1}\left[\exp\left(\frac{60}{49}\right)\right] \\ T &= 5\cosh^{-1}\left[\exp\left(\frac{60}{49}\right)\right] \approx 9.48 \text{ s} \end{aligned}$$