Problem 12

A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount currently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.

(a) If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.

(b) Find an expression for the amount of thorium-234 present at any time t.

(c) Find the time required for the thorium-234 to decay to one-half its original amount.

Solution

$$Q' = -rQ$$

Divide both sides by Q.

$$\frac{Q'}{Q} = -r$$

The left side can be written as $d/dt(\ln Q)$ by the chain rule.

$$\frac{d}{dt}\ln Q = -r$$

Integrate both sides with respect to t.

$$\ln Q = -rt + C$$

Exponentiate both sides.

$$\begin{aligned} Q(t) &= e^{-rt+C} \\ &= e^C e^{-rt} \end{aligned}$$

Let $A = e^C$.

$$Q(t) = Ae^{-rt}$$

Suppose there is a mass of Q_0 initially so that the initial condition is $Q(0) = Q_0$. Then the constant A can be determined.

$$Q(0) = A = Q_0$$

The mass at any time is thus

$$Q(t) = Q_0 e^{-rt}. \tag{1}$$

Part (a)

In order to determine r, set $Q_0 = 100$ and $Q = 82.04$ and $t = 7$ days and solve the resulting equation for it.

$$\begin{aligned} 82.04 &= 100e^{-7r} \\ e^{-7r} &= 0.8204 \\ \ln e^{-7r} &= \ln 0.8204 \\ -7r \ln e &= \ln 0.8204 \\ -7r &= \ln 0.8204 \end{aligned}$$

Therefore,

$$r = -\frac{\ln 0.8204}{7}\,\frac{1}{\text{day}} \approx 0.02828\,\frac{1}{\text{day}}.$$

r has units of 1/day because the exponent of e must be dimensionless.

Part (b)

As a result, equation (1) becomes

$$Q(t) = Q_0 e^{-0.02828t}.$$

Part (c)

In order to find the time required for the thorium-234 to decay to one-half its original amount, set $Q = Q_0/2$ and solve the resulting equation for t.

$$\begin{aligned}
\frac{Q_0}{2} &= Q_0 e^{-0.02828t} \\
e^{-0.02828t} &= \frac{1}{2} \\
\ln e^{-0.02828t} &= \ln \frac{1}{2} \\
-0.02828t \ln e &= -\ln 2 \\
0.02828t &= \ln 2 \\
t &= \frac{\ln 2}{0.02828} \text{ days} \approx 24.51 \text{ days}
\end{aligned}$$

This time is known as the half-life of thorium-234.