Problem 19

Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of a nontoxic dye that leaves a swimmer's skin an unattractive green. The pool's filtering system can take water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 gal/min.

(a) Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at any time t.

(b) Solve the problem in part (a).

(c) You have invited several dozen friends to a pool party that is scheduled to begin in 4 h. You have also determined that the effect of the dye is imperceptible if its concentration is less than 0.02 g/gal. Is your filtering system capable of reducing the dye concentration to this level within 4 h?

(d) Find the time T at which the concentration of dye first reaches the value 0.02 g/gal.

(e) Find the flow rate that is sufficient to achieve the concentration 0.02 g/gal within 4 h.

Solution

Part (a)

Apply the law of conservation of mass, which states that matter is neither created nor destroyed. If the dye flows out of the pool at some rate, then the amount in the pool will diminish. This idea is expressed mathematically as follows.

$$\text{rate of dye accumulation} = -\text{rate of dye out}$$

Let $q = q(t)$ denote the dye mass in grams and let t denote the time in minutes. The rates will then be in units of grams per minute.

$$\begin{aligned} \text{rate of dye accumulation} &= \frac{dq}{dt} \\ \text{rate of dye out} &= \text{dye density flowing out} \times \text{volumetric flow rate out} \\ &= \frac{q}{60\,000}\,\frac{\text{g}}{\cancel{\text{gal}}} \times 200\,\frac{\cancel{\text{gal}}}{\text{min}} = \frac{q}{300}\,\frac{\text{g}}{\text{min}} \end{aligned}$$

In writing the dye density, it has been assumed that the dye is uniformly distributed in the pool. By the law of conservation of mass then,

$$\frac{dq}{dt} = -\frac{q}{300},$$

and the initial condition associated with it is $q(0) = 5000$ because 5000 grams of dye are present initially.

Part (b)

$$q' = -\frac{q}{300}$$

Divide both sides by q.

$$\frac{q'}{q} = -\frac{1}{300}$$

The left side can be written as $d/dt(\ln q)$ by the chain rule.

$$\frac{d}{dt}\ln q = -\frac{1}{300}$$

Integrate both sides with respect to t.

$$\ln q = -\frac{1}{300}t + C$$

Exponentiate both sides.

$$\begin{aligned} q(t) &= \exp\left(-\frac{1}{300}t + C\right) \\ &= e^C \exp\left(-\frac{1}{300}t\right) \end{aligned}$$

Let $A = e^C$.

$$q(t) = A\exp\left(-\frac{1}{300}t\right)$$

Use the initial condition now to determine A.

$$q(0) = A = 5000$$

Therefore,

$$q(t) = 5000\exp\left(-\frac{1}{300}t\right).$$

Part (c)

t is in minutes, so we need to convert 4 hours to minutes.

$$4\,\cancel{\text{hours}} \times \frac{60\text{ minutes}}{1\,\cancel{\text{hour}}} = 240\text{ minutes}$$

The dye concentration in the pool after 4 hours is

$$\frac{q(240)}{60\,000} = \frac{1}{12e^{4/5}} \approx 0.0374\,\frac{\text{g}}{\text{gal}},$$

assuming the dye is uniformly distributed. This is not less than 0.02 g/gal, so the filtering system is not capable of reducing the dye concentration in time.

Part (d)

Set the dye concentration equal to 0.02 and solve the resulting equation for T.

$$\begin{aligned}
\frac{q(T)}{60\,000} &= 0.02 \\
\frac{1}{12}\exp\left(-\frac{1}{300}T\right) &= 0.02 \\
\exp\left(-\frac{1}{300}T\right) &= 0.24 \\
\ln\exp\left(-\frac{1}{300}T\right) &= \ln 0.24 \\
-\frac{1}{300}T &= \ln 0.24 \\
T = -300\ln 0.24 &\approx 428.1 \text{ min} \approx 7.14 \text{ hours}
\end{aligned}$$

Part (e)

The volumetric flow rate is now unknown.

$$\begin{aligned}
\text{rate of dye out} &= \text{dye density flowing out} \times \text{volumetric flow rate out} \\
&= \frac{q}{60\,000}\,\frac{\text{g}}{\cancel{\text{gal}}} \times x\,\frac{\cancel{\text{gal}}}{\text{min}} = \frac{qx}{60\,000}\,\frac{\text{g}}{\text{min}}
\end{aligned}$$

The initial value problem to solve is then

$$\frac{dq}{dt} = -\frac{qx}{60\,000}, \quad q(0) = 5000.$$

Its solution is the same as before, but here the exponent of e has changed.

$$q(t) = 5000\exp\left(-\frac{x}{60\,000}t\right).$$

We require that the dye concentration is 0.02 g/gal at 4 hours. Solve the resulting equation for x.

$$\begin{aligned}
\frac{q(240)}{60\,000} &= 0.02 \\
\frac{1}{12}\exp\left(-\frac{x}{250}\right) &= 0.02 \\
\exp\left(-\frac{x}{250}\right) &= 0.24 \\
\ln\exp\left(-\frac{x}{250}\right) &= \ln 0.24 \\
-\frac{x}{250} &= \ln 0.24 \\
x = -250\ln 0.24 &\approx 356.8\ \frac{\text{gal}}{\text{min}}
\end{aligned}$$