

## Problem 1

Solve each of the following initial value problems and plot the solutions for several values of  $y_0$ . Then describe in a few words how the solutions resemble, and differ from, each other.

- (a)  $dy/dt = -y + 5, \quad y(0) = y_0$   
(b)  $dy/dt = -2y + 5, \quad y(0) = y_0$   
(c)  $dy/dt = -2y + 10, \quad y(0) = y_0$
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### Solution

#### Part (a)

$$\begin{aligned}y' &= -y + 5 \\ &= -(y - 5)\end{aligned}$$

Divide both sides by  $y - 5$ .

$$\frac{y'}{y - 5} = -1$$

The left side can be written as  $d/dt(\ln|y - 5|)$  by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt} \ln|y - 5| = -1$$

Integrate both sides with respect to  $t$ .

$$\ln|y - 5| = -t + C$$

Exponentiate both sides.

$$\begin{aligned}|y - 5| &= e^{-t+C} \\ &= e^C e^{-t}\end{aligned}$$

Introduce  $\pm$  on the right side in order to remove the absolute value sign.

$$y - 5 = \pm e^C e^{-t}$$

Let  $A = \pm e^C$  and add 5 to both sides to solve for  $y$ .

$$y(t) = 5 + Ae^{-t}$$

Apply the initial condition now to determine  $A$ .

$$y(0) = 5 + A = y_0 \quad \rightarrow \quad A = y_0 - 5$$

Therefore,

$$y(t) = 5 + (y_0 - 5)e^{-t}.$$

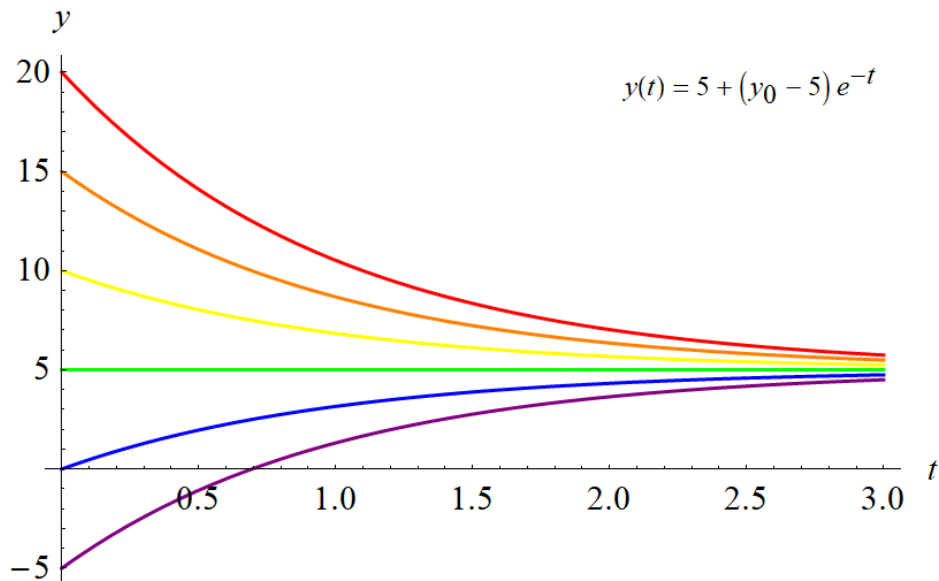


Figure 1: In this figure  $y(t)$  is plotted versus  $t$  for various values of  $y_0$ . The curves in red, orange, yellow, green, blue, and purple correspond to  $y_0 = 20$ ,  $y_0 = 15$ ,  $y_0 = 10$ ,  $y_0 = 5$ ,  $y_0 = 0$ , and  $y_0 = -5$ , respectively. The equilibrium solution is at  $y = 5$ , and the nonequilibrium solutions converge to it more slowly than in (b) and (c).

### Part (b)

$$\begin{aligned} y' &= -2y + 5 \\ &= -2 \left( y - \frac{5}{2} \right) \end{aligned}$$

Divide both sides by  $y - \frac{5}{2}$ .

$$\frac{y'}{y - \frac{5}{2}} = -2$$

The left side can be written as  $d/dt(\ln|y - 5/2|)$  by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt} \ln \left| y - \frac{5}{2} \right| = -2$$

Integrate both sides with respect to  $t$ .

$$\ln \left| y - \frac{5}{2} \right| = -2t + C$$

Exponentiate both sides.

$$\begin{aligned} \left| y - \frac{5}{2} \right| &= e^{-2t+C} \\ &= e^C e^{-2t} \end{aligned}$$

Introduce  $\pm$  on the right side in order to remove the absolute value sign.

$$y - \frac{5}{2} = \pm e^C e^{-2t}$$

Let  $A = \pm e^C$  and add  $5/2$  to both sides to solve for  $y$ .

$$y(t) = \frac{5}{2} + Ae^{-2t}$$

Apply the initial condition now to determine  $A$ .

$$y(0) = \frac{5}{2} + A = y_0 \quad \rightarrow \quad A = y_0 - \frac{5}{2}$$

Therefore,

$$y(t) = \frac{5}{2} + \left(y_0 - \frac{5}{2}\right) e^{-2t}.$$

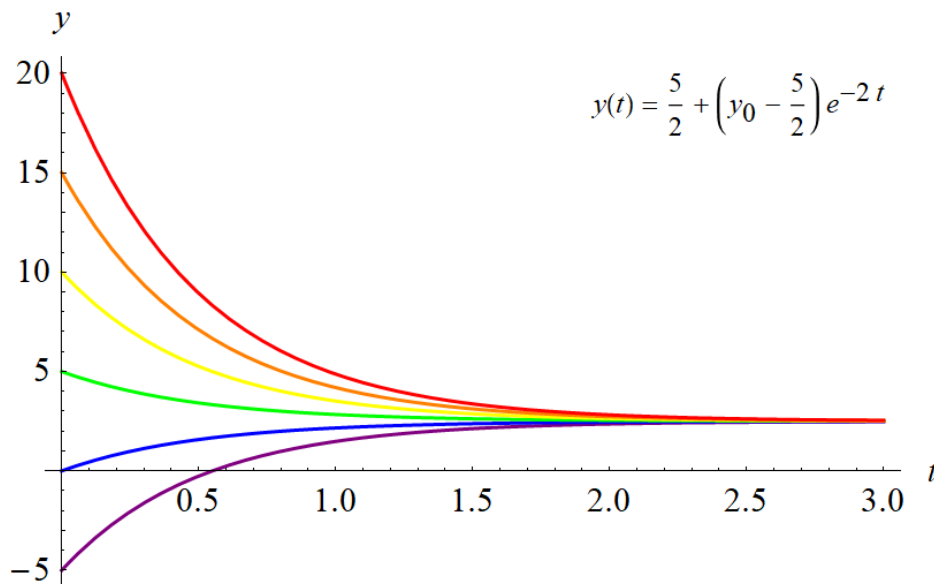


Figure 2: In this figure  $y(t)$  is plotted versus  $t$  for various values of  $y_0$ . The curves in red, orange, yellow, green, blue, and purple correspond to  $y_0 = 20$ ,  $y_0 = 15$ ,  $y_0 = 10$ ,  $y_0 = 5$ ,  $y_0 = 0$ , and  $y_0 = -5$ , respectively. The equilibrium solution is at  $y = 5/2$ , and the nonequilibrium solutions converge to it faster than in (a).

### Part (c)

$$\begin{aligned} y' &= -2y + 10 \\ &= -2(y - 5) \end{aligned}$$

Divide both sides by  $y - 5$ .

$$\frac{y'}{y - 5} = -2$$

The left side can be written as  $d/dt(\ln |y - 5|)$  by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt} \ln |y - 5| = -2$$

Integrate both sides with respect to  $t$ .

$$\ln |y - 5| = -2t + C$$

Exponentiate both sides.

$$\begin{aligned} |y - 5| &= e^{-2t+C} \\ &= e^C e^{-2t} \end{aligned}$$

Introduce  $\pm$  on the right side in order to remove the absolute value sign.

$$y - 5 = \pm e^C e^{-2t}$$

Let  $A = \pm e^C$  and add 5 to both sides to solve for  $y$ .

$$y(t) = 5 + A e^{-2t}$$

Apply the initial condition now to determine  $A$ .

$$y(0) = 5 + A = y_0 \quad \rightarrow \quad A = y_0 - 5$$

Therefore,

$$y(t) = 5 + (y_0 - 5)e^{-2t}.$$

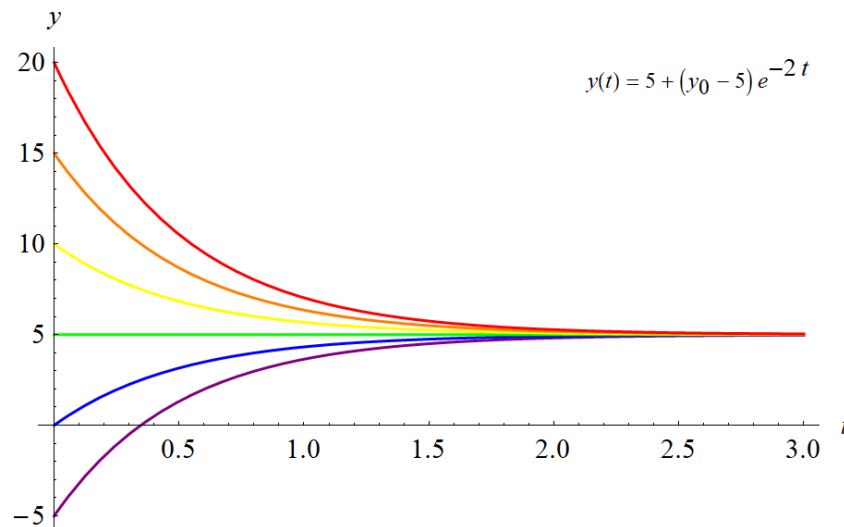


Figure 3: In this figure  $y(t)$  is plotted versus  $t$  for various values of  $y_0$ . The curves in red, orange, yellow, green, blue, and purple correspond to  $y_0 = 20$ ,  $y_0 = 15$ ,  $y_0 = 10$ ,  $y_0 = 5$ ,  $y_0 = 0$ , and  $y_0 = -5$ , respectively. The equilibrium solution is at  $y = 5$ , and the nonequilibrium solutions converge to it faster than in (a).