Problem 18

A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical (see Problem 21 of Section 1.1). Water containing 0.01 g/gal of the chemical flows into the pond at a rate of 300 gal/h, and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond.

(a) Let $Q(t)$ be the amount of the chemical in the pond at time t. Write down an initial value problem for $Q(t)$.

(b) Solve the problem in part (a) for $Q(t)$. How much chemical is in the pond after 1 year?

(c) At the end of 1 year the source of the chemical in the pond is removed; thereafter pure water flows into the pond, and the mixture flows out at the same rate as before. Write down the initial value problem that describes this new situation.

(d) Solve the initial value problem in part (c). How much chemical remains in the pond after 1 additional year (2 years from the beginning of the problem)?

(e) How long does it take for $Q(t)$ to be reduced to 10 g?

(f) Plot $Q(t)$ versus t for 3 years.

Solution

Part (a)

Apply the law of conservation of mass, which states that matter is neither created nor destroyed. If the chemical flows into the pond at some rate, then it must flow out at that same rate or else the chemical will accumulate in the pond (assuming it flows in faster). This idea is expressed mathematically as follows.

$$\text{rate of chemical accumulation} = \text{rate of chemical in} - \text{rate of chemical out}$$

Let $Q = Q(t)$ denote the chemical mass in grams and let t denote the time in hours. The rates will then be in units of grams per hour.

$$\begin{aligned}
\text{rate of chemical accumulation} &= \frac{dQ}{dt} \\
\text{rate of chemical in} &= \text{chemical density flowing in} \times \text{volumetric flow rate in} \\
&= 0.01\ \frac{\text{g}}{\cancel{\text{gal}}} \times 300\ \frac{\cancel{\text{gal}}}{\text{h}} = 3\ \frac{\text{g}}{\text{h}} \\
\text{rate of chemical out} &= \text{chemical density flowing out} \times \text{volumetric flow rate out} \\
&= \frac{Q}{1\,000\,000}\ \frac{\text{g}}{\cancel{\text{gal}}} \times 300\ \frac{\cancel{\text{gal}}}{\text{h}} = \frac{3Q}{10\,000}\ \frac{\text{g}}{\text{h}}
\end{aligned}$$

The assumption that the chemical is uniformly distributed in the pond allows us to write the chemical density flowing out as we have. By the law of conservation of mass then,

$$\frac{dQ}{dt} = 3 - \frac{3Q}{10\,000},$$

and the initial condition associated with it is $Q(0) = 0$ because no chemical is present initially.

Part (b)

$$\begin{aligned} Q' &= 3 - \frac{3Q}{10\,000} \\ &= -\frac{3}{10\,000}(Q - 10\,000) \end{aligned}$$

Divide both sides by $Q - 10\,000$.

$$\frac{Q'}{Q - 10\,000} = -\frac{3}{10\,000}$$

The left side can be written as $d/dt(\ln|Q - 10\,000|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}\ln|Q - 10\,000| = -\frac{3}{10\,000}$$

Integrate both sides with respect to t.

$$\ln|Q - 10\,000| = -\frac{3}{10\,000}t + C_1$$

Exponentiate both sides.

$$\begin{aligned} |Q - 10\,000| &= \exp\left(-\frac{3}{10\,000}t + C_1\right) \\ &= e^{C_1}\exp\left(-\frac{3}{10\,000}t\right) \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$Q(t) - 10\,000 = \pm e^{C_1}\exp\left(-\frac{3}{10\,000}t\right)$$

Let $A = \pm e^{C_1}$ and add 10 000 to both sides to solve for $Q(t)$.

$$Q(t) = 10\,000 + A\exp\left(-\frac{3}{10\,000}t\right)$$

Apply the initial condition here to determine A.

$$Q(0) = 10\,000 + A = 0 \quad \rightarrow \quad A = -10\,000$$

Therefore,

$$\begin{aligned} Q(t) &= 10\,000 - 10\,000\exp\left(-\frac{3}{10\,000}t\right) \\ &= 10\,000\left[1 - \exp\left(-\frac{3}{10\,000}t\right)\right]. \end{aligned}$$

t is in hours, so we need to convert 1 year to hours.

$$1\ \cancel{\text{year}} \times \frac{365\ \cancel{\text{days}}}{1\ \cancel{\text{year}}} \times \frac{24\text{ hours}}{1\ \cancel{\text{day}}} = 8760\text{ hours}$$

As a result,

$$Q(8760) = 10\,000\left[1 - \exp\left(-\frac{657}{250}\right)\right] \approx 9277.77\text{ g.}$$

Part (c)

The rate of chemical in is now zero, so the initial value problem is

$$\frac{dQ}{dt} = -\frac{3Q}{10\,000}, \quad Q(0) = 10\,000\left[1 - \exp\left(-\frac{657}{250}\right)\right].$$

Part (d)

$$Q' = -\frac{3Q}{10\,000}$$

Divide both sides of the ODE by Q.

$$\frac{Q'}{Q} = -\frac{3}{10\,000}$$

The left side can be written as $d/dt(\ln Q)$ by the chain rule.

$$\frac{d}{dt}\ln Q = -\frac{3}{10\,000}$$

Integrate both sides with respect to t.

$$\ln Q = -\frac{3}{10\,000}t + C_2$$

Exponentiate both sides.

$$\begin{aligned} Q(t) &= \exp\left(-\frac{3}{10\,000}t + C_2\right) \\ &= e^{C_2}\exp\left(-\frac{3}{10\,000}t\right) \end{aligned}$$

Let $B = e^{C_2}$.

$$Q(t) = B\exp\left(-\frac{3}{10\,000}t\right)$$

Apply the initial condition here to determine B.

$$Q(0) = B = 10\,000\left[1 - \exp\left(-\frac{657}{250}\right)\right]$$

Therefore,

$$Q(t) = 10\,000\left[1 - \exp\left(-\frac{657}{250}\right)\right]\exp\left(-\frac{3}{10\,000}t\right).$$

After 1 year, or 8760 hours, there are

$$Q(t) = 10\,000\left[1 - \exp\left(-\frac{657}{250}\right)\right]\exp\left(-\frac{657}{250}\right) \approx 670.07 \text{ g}$$

left in the pond.

Part (e)

To find how long it takes for $Q(t)$ to be reduced to 10 g, set $Q = 10$ and solve the resulting equation for t.

$$\begin{aligned}
10 &= 10\,000\left[1-\exp\left(-\frac{657}{250}\right)\right]\exp\left(-\frac{3}{10\,000}t\right)\\
\frac{1}{1000} &= \left[1-\exp\left(-\frac{657}{250}\right)\right]\exp\left(-\frac{3}{10\,000}t\right)\\
\frac{1}{1000\left[1-\exp\left(-\frac{657}{250}\right)\right]} &= \exp\left(-\frac{3}{10\,000}t\right)\\
\ln\frac{1}{1000\left[1-\exp\left(-\frac{657}{250}\right)\right]} &= \ln\exp\left(-\frac{3}{10\,000}t\right)\\
\ln\frac{1}{1000\left[1-\exp\left(-\frac{657}{250}\right)\right]} &= -\frac{3}{10\,000}t\\
t = -\frac{10\,000}{3}\ln\frac{1}{1000\left[1-\exp\left(-\frac{657}{250}\right)\right]} &\approx 22\,776 \text{ hours} \approx 2.6 \text{ years}
\end{aligned}$$

Part (f)

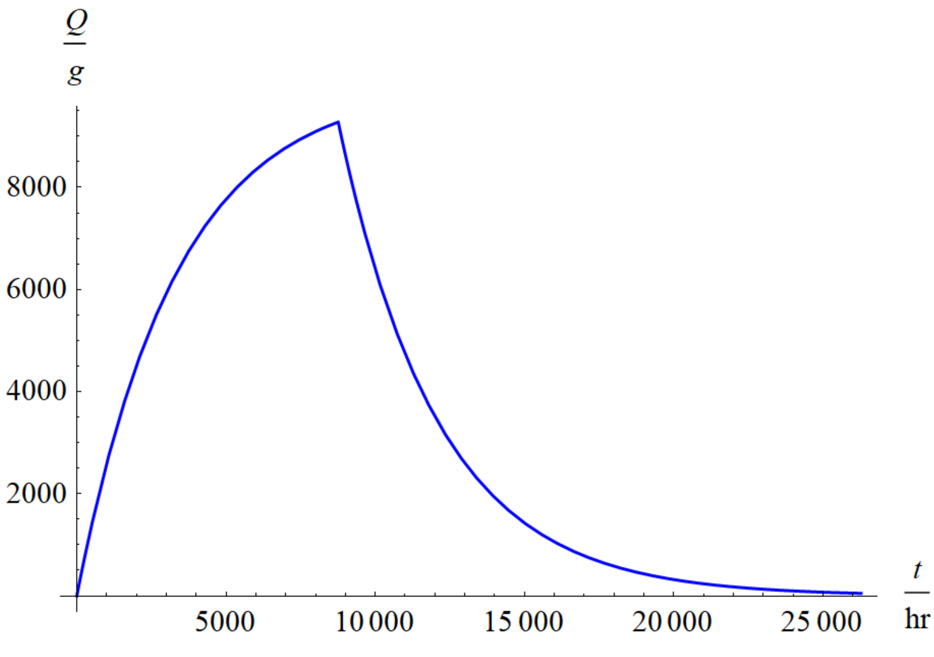

Figure 1: This figure shows a plot of $Q(t)$ versus time.