Problem 5

Undetermined coefficients. Here is an alternative way to solve the equation

$$dy/dt = ay - b. \tag{i}$$

(a) Solve the simpler equation

$$dy/dt = ay. \tag{ii}$$

Call the solution $y_1(t)$.

(b) Observe that the only difference between Eqs. (i) and (ii) is the constant $-b$ in Eq. (i). Therefore, it may seem reasonable to assume that the solutions of these two equations also differ only by a constant. Test this assumption by trying to find a constant k such that $y = y_1(t) + k$ is a solution of Eq. (i).

(c) Compare your solution from part (b) with the solution given in the text in Eq. (17).

Note: This method can also be used in some cases in which the constant b is replaced by a function $g(t)$. It depends on whether you can guess the general form that the solution is likely to take. This method is described in detail in Section 3.5 in connection with second order equations.

Solution

Part (a)

$$y' = ay$$

Divide both sides by y.

$$\frac{y'}{y} = a$$

The left side can be written as $d/dt(\ln|y|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}\ln|y| = a$$

Integrate both sides with respect to t.

$$\ln|y| = at + C$$

Exponentiate both sides.

$$\begin{aligned} |y| &= e^{at+C} \\ &= e^C e^{at} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$y(t) = \pm e^C e^{at}$$

Use a new constant A for the number in front of the exponential function.

$$y(t) = Ae^{at}$$

Therefore,

$$y_1(t) = Ae^{at}.$$

Part (b)

Suppose that

$$\begin{aligned} y(t) &= y_1(t) + k \\ &= Ae^{at} + k. \end{aligned}$$

Substitute this into equation (i).

$$\frac{dy}{dt} = ay - b \quad \rightarrow \quad \begin{aligned} \frac{d}{dt}(Ae^{at} + k) &= a(Ae^{at} + k) - b \\ Aae^{at} &= Aae^{at} + ak - b \\ 0 &= ak - b \\ k &= \frac{b}{a} \end{aligned}$$

Part (c)

The solution for y obtained with the method of undetermined coefficients is

$$y(t) = Ae^{at} + \frac{b}{a}.$$

Equation (17) in the text is

$$y = (b/a) + ce^{at}, \tag{17}$$

which is the same.