Problem 6

Use the method of Problem 5 to solve the equation

$$dy/dt = -ay + b.$$

Solution

Start by solving the associated homogeneous equation.

$$y_1' = -ay_1$$

Divide both sides by y_1.

$$\frac{y_1'}{y_1} = -a$$

The left side can be written as $d/dt(\ln|y_1|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}\ln|y_1| = -a$$

Integrate both sides with respect to t.

$$\ln|y_1| = -at + C$$

Exponentiate both sides.

$$\begin{aligned}|y_1| &= e^{-at+C} \\ &= e^C e^{-at}\end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$y_1(t) = \pm e^C e^{-at}$$

Use a new constant A for the number in front of the exponential function.

$$y_1(t) = Ae^{-at}$$

Because b is a constant, we postulate that the solution to the original ODE is

$$y(t) = y_1(t) + k,$$

where k is a constant to be determined. Substitute this formula into the original ODE to find it.

$$\frac{dy}{dt} = -ay + b \quad \rightarrow \quad \begin{aligned}\frac{d}{dt}(Ae^{-at} + k) &= -a(Ae^{-at} + k) + b \\ -Aae^{-at} &= -Aae^{-at} - ak + b \\ 0 &= -ak + b \\ k &= \frac{b}{a}\end{aligned}$$

Therefore,

$$y(t) = Ae^{-at} + \frac{b}{a}.$$