Problem 8

Consider a population p of field mice that grows at a rate proportional to the current population, so that $dp/dt = rp$.

(a) Find the rate constant r if the population doubles in 30 days.

(b) Find r if the population doubles in N days.

Solution

Note that t is measured in days in this problem.

$$p' = rp$$

Divide both sides by p.

$$\frac{p'}{p} = r$$

The left side can be written as $d/dt(\ln|p|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}\ln|p| = r$$

Integrate both sides with respect to t.

$$\ln|p| = rt + C$$

Exponentiate both sides.

$$\begin{aligned}|p| &= e^{rt+C}\\ &= e^C e^{rt}\end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$p(t) = \pm e^C e^{rt}$$

Let $A = \pm e^C$.

$$p(t) = Ae^{rt}$$

Apply the initial condition $p(0) = p_0$, where p_0 is the initial mouse population, to determine A.

$$p(0) = A = p_0$$

Therefore, the general solution to the ODE is

$$p(t) = p_0 e^{rt}.$$

Part (a)

In order to find the rate constant r if the population doubles in 30 days, set $p = 2p_0$ and $t = 30$ days and solve the resulting equation for r.

$$\begin{aligned} 2p_0 &= p_0 e^{30r} \\ 2 &= e^{30r} \\ \ln 2 &= \ln e^{30r} \\ \ln 2 &= 30r \ln e \\ \ln 2 &= 30r \end{aligned}$$

Therefore,

$$r = \frac{\ln 2}{30} \approx 0.0231 \, \frac{1}{\text{day}}.$$

r is in units of 1/day because the exponent of e must be dimensionless.

Part (b)

In order to find the rate constant r if the population doubles in N days, set $p = 2p_0$ and $t = N$ and solve the resulting equation for r.

$$\begin{aligned} 2p_0 &= p_0 e^{rN} \\ 2 &= e^{rN} \\ \ln 2 &= \ln e^{rN} \\ \ln 2 &= rN \ln e \\ \ln 2 &= rN \end{aligned}$$

Therefore,

$$r = \frac{\ln 2}{N}.$$