

Problem 28

In each of Problems 25 through 28, verify that each given function is a solution of the given partial differential equation.

$$\alpha^2 u_{xx} = u_t; \quad u(x, t) = (\pi/t)^{1/2} e^{-x^2/4\alpha^2 t}, \quad t > 0$$

Solution

$$\begin{aligned} \alpha^2 \frac{\partial^2 u}{\partial x^2} &\stackrel{?}{=} \frac{\partial u}{\partial t} \\ \alpha^2 \frac{\partial^2}{\partial x^2} \left[\sqrt{\frac{\pi}{t}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \right] &\stackrel{?}{=} \frac{\partial}{\partial t} \left[\sqrt{\frac{\pi}{t}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \right] \\ \alpha^2 \sqrt{\frac{\pi}{t}} \frac{\partial}{\partial x} \left[\left(-\frac{x}{2\alpha^2 t}\right) \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \right] &\stackrel{?}{=} -\frac{1}{2} \sqrt{\frac{\pi}{t^3}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) + \sqrt{\frac{\pi}{t}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \left(\frac{x^2}{4\alpha^2 t^2}\right) \\ -\frac{1}{2t} \sqrt{\frac{\pi}{t}} \frac{\partial}{\partial x} \left[x \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \right] &\stackrel{?}{=} \left(-\frac{1}{2} + \frac{x^2}{4\alpha^2 t}\right) \sqrt{\frac{\pi}{t^3}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \\ -\frac{1}{2t} \sqrt{\frac{\pi}{t}} \left[\exp\left(-\frac{x^2}{4\alpha^2 t}\right) + x \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \left(-\frac{x}{2\alpha^2 t}\right) \right] &\stackrel{?}{=} \left(-\frac{1}{2} + \frac{x^2}{4\alpha^2 t}\right) \sqrt{\frac{\pi}{t^3}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \\ -\frac{1}{2t} \sqrt{\frac{\pi}{t}} \left(1 - \frac{x^2}{2\alpha^2 t}\right) \exp\left(-\frac{x^2}{4\alpha^2 t}\right) &\stackrel{?}{=} \left(-\frac{1}{2} + \frac{x^2}{4\alpha^2 t}\right) \sqrt{\frac{\pi}{t^3}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \\ \frac{1}{t} \sqrt{\frac{\pi}{t}} \left(-\frac{1}{2} + \frac{x^2}{4\alpha^2 t}\right) \exp\left(-\frac{x^2}{4\alpha^2 t}\right) &\stackrel{?}{=} \left(-\frac{1}{2} + \frac{x^2}{4\alpha^2 t}\right) \sqrt{\frac{\pi}{t^3}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \\ \left(-\frac{1}{2} + \frac{x^2}{4\alpha^2 t}\right) \sqrt{\frac{\pi}{t^3}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) &= \left(-\frac{1}{2} + \frac{x^2}{4\alpha^2 t}\right) \sqrt{\frac{\pi}{t^3}} \exp\left(-\frac{x^2}{4\alpha^2 t}\right) \end{aligned}$$

The solution is verified.