Problem 20

In each of Problems 19 and 20, determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

$$t^2 y'' - 4ty' + 4y = 0$$

Solution

Because the ODE is equidimensional, the solution is of the form $y = t^r$. Substitute it into the equation to determine r.

$$\begin{aligned} t^2 y'' - 4ty' + 4y &= 0 \\ t^2 \frac{d^2}{dt^2}(t^r) - 4t \frac{d}{dt}(t^r) + 4(t^r) &= 0 \\ t^2 r(r-1)t^{r-2} - 4trt^{r-1} + 4t^r &= 0 \\ r(r-1)t^r - 4rt^r + 4t^r &= 0 \end{aligned}$$

Divide both sides by t^r.

$$r(r-1) - 4r + 4 = 0$$

$$r^2 - 5r + 4 = 0$$

$$(r-1)(r-4) = 0$$

$$r = 1 \quad \text{or} \quad r = 4$$

Therefore, t and t^4 are two solutions to the ODE. The general solution is

$$y(t) = C_1 t + C_2 t^4,$$

where C_1 and C_2 are arbitrary constants.