Problem 26

In each of Problems 25 through 28, verify that each given function is a solution of the given partial differential equation.

$$\alpha^2 u_{xx} = u_t; \qquad u_1(x,t) = e^{-\alpha^2 t}\sin x, \quad u_2(x,t) = e^{-\alpha^2\lambda^2 t}\sin\lambda x, \quad \lambda \text{ a real constant}$$

Solution

$$\begin{aligned}
\alpha^2 \frac{\partial^2 u_1}{\partial x^2} &\stackrel{?}{=} \frac{\partial u_1}{\partial t} \\
\alpha^2 \frac{\partial^2}{\partial x^2}(e^{-\alpha^2 t}\sin x) &\stackrel{?}{=} \frac{\partial}{\partial t}(e^{-\alpha^2 t}\sin x) \\
\alpha^2 e^{-\alpha^2 t}(-\sin x) &\stackrel{?}{=} (-\alpha^2)e^{-\alpha^2 t}\sin x \\
-\alpha^2 e^{-\alpha^2 t}\sin x &= -\alpha^2 e^{-\alpha^2 t}\sin x
\end{aligned}$$

The first solution is verified.

$$\begin{aligned}
\alpha^2 \frac{\partial^2 u_2}{\partial x^2} &\stackrel{?}{=} \frac{\partial u_2}{\partial t} \\
\alpha^2 \frac{\partial^2}{\partial x^2}(e^{-\alpha^2\lambda^2 t}\sin\lambda x) &\stackrel{?}{=} \frac{\partial}{\partial t}(e^{-\alpha^2\lambda^2 t}\sin\lambda x) \\
\alpha^2 e^{-\alpha^2\lambda^2 t}(-\lambda^2\sin\lambda x) &\stackrel{?}{=} (-\alpha^2\lambda^2)e^{-\alpha^2\lambda^2 t}\sin\lambda x \\
-\alpha^2\lambda^2 e^{-\alpha^2\lambda^2 t}\sin\lambda x &= -\alpha^2\lambda^2 e^{-\alpha^2\lambda^2 t}\sin\lambda x
\end{aligned}$$

The second solution is verified.