Problem 4

In each of Problems 1 through 12:

(a) Draw a direction field for the given differential equation.

(b) Based on an inspection of the direction field, describe how solutions behave for large t.

(c) Find the general solution of the given differential equation, and use it to determine how solutions behave as $t \to \infty$.

$$y' + (1/t)y = 3\cos 2t, \qquad t > 0$$

Solution

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dt, dy \rangle = \left\langle 1, \frac{dy}{dt} \right\rangle dt = \langle 1, -\frac{y}{t} + 3\cos 2t \rangle \, dt$$

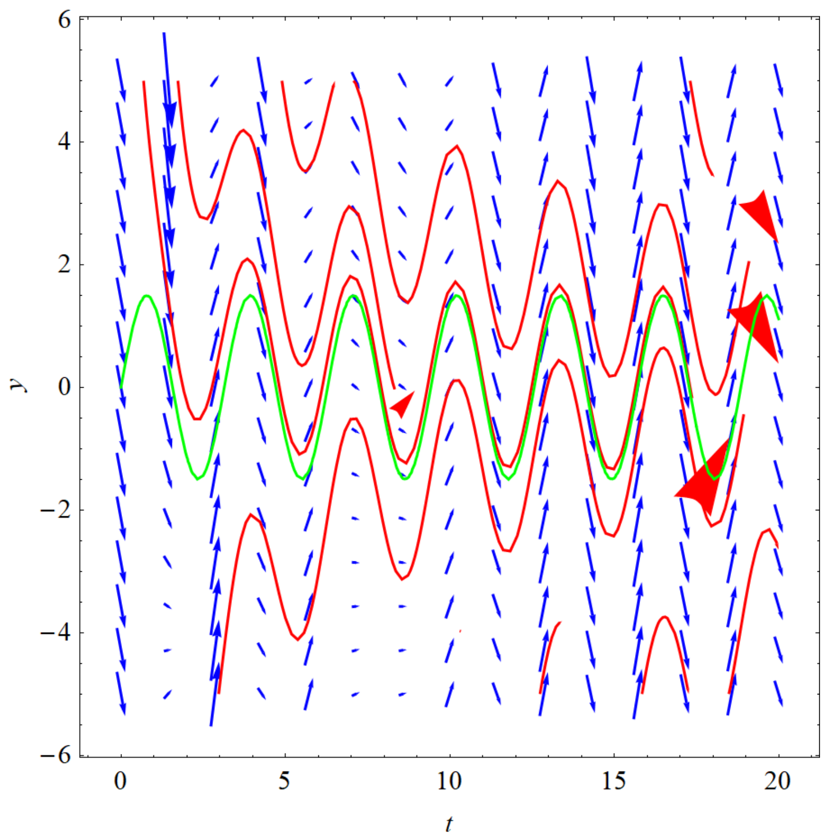

Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. In green is the asymptotic solution $y = (3/2)\sin 2t$ that all solutions tend to as $t \to \infty$.

$$y' + \frac{y}{t} = 3\cos 2t$$

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left(\int^t \frac{1}{s}\,ds\right) = e^{\ln t} = t$$

Multiply both sides of the ODE by $I(t)$.

$$ty' + y = 3t\cos 2t$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(ty) = 3t\cos 2t$$

Integrate both sides with respect to t.

$$\begin{aligned} ty &= \int^t 3s\cos 2s\,ds + C \\ &= \frac{3}{4}\cos 2t + \frac{3}{2}t\sin 2t + C \end{aligned}$$

Divide both sides by t to solve for $y(t)$.

$$y(t) = \frac{3}{2}\sin 2t + \frac{1}{t}\left(\frac{3}{4}\cos 2t + C\right)$$

Therefore, the asymptotic solution is

$$\lim_{t\to\infty} y(t) = \frac{3}{2}\sin 2t.$$