Problem 14

In each of Problems 13 through 20, find the solution of the given initial value problem.

$$y' + 2y = te^{-2t}, \qquad y(1) = 0$$

Solution

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left(\int^t 2\,ds\right) = e^{2t}$$

Multiply both sides of the ODE by $I(t)$.

$$e^{2t}y' + 2e^{2t}y = t$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(e^{2t}y) = t$$

Integrate both sides with respect to t.

$$e^{2t}y = \frac{t^2}{2} + C$$

Divide both sides by e^{2t} to solve for $y(t)$.

$$y(t) = e^{-2t}\left(\frac{t^2}{2} + C\right)$$

Apply the initial condition $y(1) = 0$ here to determine C.

$$y(1) = e^{-2}\left(\frac{1}{2} + C\right) = 0 \quad \rightarrow \quad C = -\frac{1}{2}$$

Therefore,

$$\begin{aligned} y(t) &= e^{-2t}\left(\frac{t^2}{2} - \frac{1}{2}\right) \\ &= \frac{e^{-2t}}{2}(t^2 - 1). \end{aligned}$$