Problem 15

In each of Problems 13 through 20, find the solution of the given initial value problem.

$$ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2}, \quad t > 0$$

Solution

Start by dividing both sides by t to make the coefficient of y' 1.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left(\int^t \frac{2}{s}\,ds\right) = e^{2\ln t} = e^{\ln t^2} = t^2$$

Multiply both sides of the ODE by $I(t)$.

$$t^2y' + 2ty = t^3 - t^2 + t$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(t^2y) = t^3 - t^2 + t$$

Integrate both sides with respect to t.

$$t^2y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + C$$

Divide both sides by t^2 to solve for $y(t)$.

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{C}{t^2}$$

Apply the initial condition $y(1) = 1/2$ here to determine C.

$$y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + \frac{C}{1} = \frac{1}{2} \quad \rightarrow \quad C = \frac{1}{12}$$

Therefore,

$$y(t) = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{1}{12t^2}.$$