Problem 16

In each of Problems 13 through 20, find the solution of the given initial value problem.

$$y' + (2/t)y = (\cos t)/t^2, \qquad y(\pi) = 0, \quad t > 0$$

Solution

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2}$$

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left(\int^t \frac{2}{s}\,ds\right) = e^{2\ln t} = e^{\ln t^2} = t^2$$

Multiply both sides of the ODE by $I(t)$.

$$t^2y' + 2ty = \cos t$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(t^2y) = \cos t$$

Integrate both sides with respect to t.

$$t^2y = \sin t + C$$

Divide both sides by t^2 to solve for $y(t)$.

$$y(t) = \frac{\sin t + C}{t^2}$$

Apply the initial condition $y(\pi) = 0$ here to determine C.

$$y(\pi) = \frac{0 + C}{\pi^2} = 0 \quad \to \quad C = 0$$

Therefore,

$$y(t) = \frac{\sin t}{t^2}.$$