Problem 29

Consider the initial value problem

$$y' + \frac{1}{4}y = 3 + 2\cos 2t, \qquad y(0) = 0.$$

(a) Find the solution of this initial value problem and describe its behavior for large t.

(b) Determine the value of t for which the solution first intersects the line $y = 12$.

Solution

$$y' + \frac{1}{4}y = 3 + 2\cos 2t$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t \frac{1}{4}\,ds\right) = e^{t/4}$$

Proceed with the multiplication.

$$e^{t/4}y' + \frac{1}{4}e^{t/4}y = e^{t/4}(3 + 2\cos 2t)$$

The left side can be written as $d/dt(Iy)$ using the product rule.

$$\frac{d}{dt}(e^{t/4}y) = e^{t/4}(3 + 2\cos 2t)$$

Integrate both sides with respect to t.

$$\begin{aligned} e^{t/4}y &= \int^t e^{s/4}(3 + 2\cos 2s)\,ds + C \\ &= \int^t 3e^{s/4}\,ds + \int^t 2e^{s/4}\cos 2s\,ds + C \\ &= 3(4)e^{t/4} + 2\int^t e^{s/4}\cos 2s\,ds + C \end{aligned} \tag{1}$$

Evaluate the remaining integral on the right side by integrating by parts twice.

$$\begin{aligned} \int^t e^{s/4}\cos 2s\,ds &= \int^t e^{s/4}\frac{d}{ds}\left(\frac{1}{2}\sin 2s\right)ds \\ &= e^{t/4}\left(\frac{1}{2}\sin 2t\right) - \int^t \left(\frac{1}{4}\right)e^{s/4}\left(\frac{1}{2}\sin 2s\right)ds \\ &= \frac{1}{2}e^{t/4}\sin 2t - \frac{1}{8}\int^t e^{s/4}\sin 2s\,ds \\ &= \frac{1}{2}e^{t/4}\sin 2t - \frac{1}{8}\int^t e^{s/4}\frac{d}{ds}\left(-\frac{1}{2}\cos 2s\right)ds \\ &= \frac{1}{2}e^{t/4}\sin 2t - \frac{1}{8}\left[e^{t/4}\left(-\frac{1}{2}\cos 2t\right) - \int^t \left(\frac{1}{4}\right)e^{s/4}\left(-\frac{1}{2}\cos 2s\right)ds\right] \end{aligned}$$

Simplify the right side.

$$\int^t e^{s/4}\cos 2s\,ds = \frac{1}{2}e^{t/4}\sin 2t + \frac{1}{16}e^{t/4}\cos 2t - \frac{1}{64}\int^t e^{s/4}\cos 2s\,ds$$

Solve this equation for the desired integral.

$$\frac{65}{64}\int^t e^{s/4}\cos 2s\,ds = \frac{1}{2}e^{t/4}\sin 2t + \frac{1}{16}e^{t/4}\cos 2t \quad\rightarrow\quad \int^t e^{s/4}\cos 2s\,ds = \frac{4}{65}e^{t/4}(8\sin 2t + \cos 2t)$$

Substitute this result into equation (1).

$$\begin{aligned} e^{t/4}y &= 3(4)e^{t/4} + 2\left[\frac{4}{65}e^{t/4}(8\sin 2t + \cos 2t)\right] + C \\ &= 12e^{t/4} + \frac{8}{65}e^{t/4}(8\sin 2t + \cos 2t) + C \end{aligned}$$

Divide both sides by $e^{t/4}$ to obtain the general solution for y.

$$y(t) = 12 + \frac{8}{65}(8\sin 2t + \cos 2t) + Ce^{-t/4}$$

Apply the initial condition now to determine C.

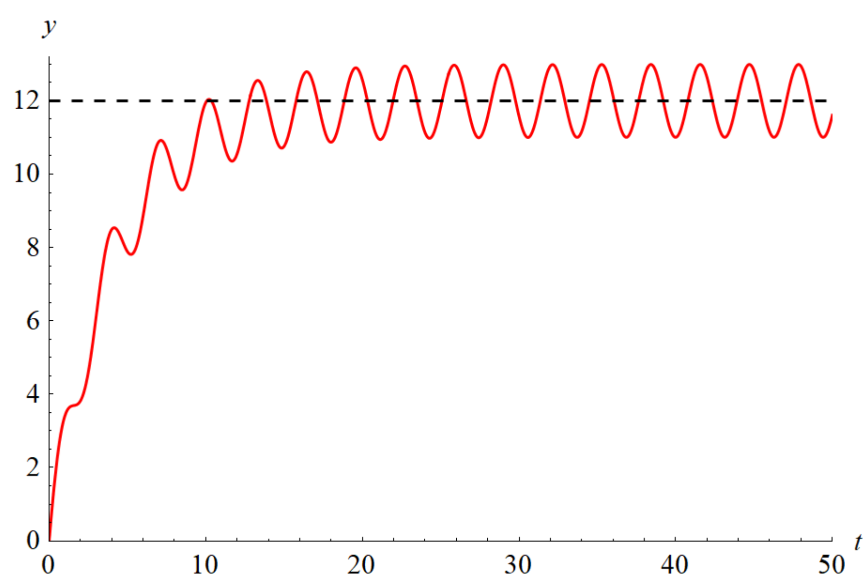
$$y(0) = 12 + \frac{8}{65}(1) + C = 0 \quad\rightarrow\quad C = -\frac{788}{65}$$

Therefore, the solution to the initial value problem is

$$y(t) = 12 + \frac{8}{65}(8\sin 2t + \cos 2t) - \frac{788}{65}e^{-t/4}.$$

Also,

$$\lim_{t\to\infty} y(t) = 12 + \frac{8}{65}(8\sin 2t + \cos 2t).$$



Judging from the graph, the solution first intersects the line $y = 12$ at $t \approx 10$.