Problem 30

Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

Solution

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^t (-1)\, ds\right] = e^{-t}$$

Proceed with the multiplication.

$$e^{-t}y' - e^{-t}y = e^{-t}(1 + 3\sin t)$$

The left side can be written as $d/dt(Iy)$ using the product rule.

$$\frac{d}{dt}(e^{-t}y) = e^{-t}(1 + 3\sin t)$$

Integrate both sides with respect to t.

$$\begin{aligned} e^{-t}y &= \int^t e^{-s}(1 + 3\sin s)\, ds + C \\ &= \int^t e^{-s}\, ds + \int^t 3e^{-s}\sin s\, ds + C \\ &= -e^{-t} + 3\int^t e^{-s}\sin s\, ds + C \qquad (1) \end{aligned}$$

Use integration by parts twice to evaluate the remaining integral.

$$\begin{aligned} \int^t e^{-s}\sin s\, ds &= \int^t e^{-s}\frac{d}{ds}(-\cos s)\, ds \\ &= e^{-t}(-\cos t) - \int^t (-1)e^{-s}(-\cos s)\, ds \\ &= -e^{-t}\cos t - \int^t e^{-s}\cos s\, ds \\ &= -e^{-t}\cos t - \int^t e^{-s}\frac{d}{ds}(\sin s)\, ds \\ &= -e^{-t}\cos t - \left[e^{-t}(\sin t) - \int^t (-1)e^{-s}(\sin s)\, ds\right] \\ &= -e^{-t}\cos t - e^{-t}\sin t - \int^t e^{-s}\sin s\, ds \end{aligned}$$

Solve the equation for the desired integral.

$$2\int^t e^{-s}\sin s\, ds = -e^{-t}\cos t - e^{-t}\sin t \quad \to \quad \int^t e^{-s}\sin s\, ds = -\frac{1}{2}e^{-t}(\cos t + \sin t)$$

Substitute this result into equation (1).

$$\begin{aligned} e^{-t}y &= -e^{-t} + 3\left[-\frac{1}{2}e^{-t}(\cos t + \sin t)\right] + C \\ &= -e^{-t} - \frac{3}{2}e^{-t}(\cos t + \sin t) + C \end{aligned}$$

Multiply both sides by e^t to obtain the general solution for y.

$$y(t) = -1 - \frac{3}{2}(\cos t + \sin t) + Ce^t$$

Now apply the initial condition to determine C.

$$y(0) = -1 - \frac{3}{2}(1) + C = y_0 \quad \rightarrow \quad C = y_0 + \frac{5}{2}$$

Therefore, the solution to the initial value problem is

$$y(t) = -1 - \frac{3}{2}(\cos t + \sin t) + \left(y_0 + \frac{5}{2}\right)e^t.$$

In order for it to remain finite as $t \to \infty$, we require the coefficient of e^t to be zero; that is,

$$y_0 = -\frac{5}{2}.$$