Problem 31

Consider the initial value problem

$$y' - \frac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

Solution

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^t \left(-\frac{3}{2}\right) ds\right] = e^{-3t/2}$$

Proceed with the multiplication.

$$e^{-3t/2}y' - \frac{3}{2}e^{-3t/2}y = e^{-3t/2}(3t + 2e^t)$$

The left side can be written as $d/dt(Iy)$ using the product rule.

$$\frac{d}{dt}(e^{-3t/2}y) = 3te^{-3t/2} + 2e^{-t/2}$$

Integrate both sides with respect to t.

$$\begin{aligned} e^{-3t/2}y &= \int^t (3se^{-3s/2} + 2e^{-s/2})\, ds + C \\ &= 3\int^t se^{-3s/2}\, ds + 2\int^t e^{-s/2}\, ds + C \\ &= 3\int^t se^{-3s/2}\, ds + 2(-2)e^{-t/2} + C \qquad (1) \end{aligned}$$

Use integration by parts to evaluate the remaining integral.

$$\begin{aligned} \int^t se^{-3s/2}\, ds &= \int^t s\frac{d}{ds}\left(-\frac{2}{3}e^{-3s/2}\right) ds \\ &= t\left(-\frac{2}{3}e^{-3t/2}\right) - \int^t \left(-\frac{2}{3}e^{-3s/2}\right) ds \\ &= -\frac{2}{3}te^{-3t/2} + \frac{2}{3}\int^t e^{-3s/2}\, ds \\ &= -\frac{2}{3}te^{-3t/2} + \frac{2}{3}\left(-\frac{2}{3}\right)e^{-3t/2} \\ &= -\frac{2}{9}e^{-3t/2}(3t + 2) \end{aligned}$$

Substitute this result into equation (1).

$$\begin{aligned} e^{-3t/2}y &= 3\left[-\frac{2}{9}e^{-3t/2}(3t + 2)\right] + 2(-2)e^{-t/2} + C \\ &= -\frac{2}{3}e^{-3t/2}(3t + 2) - 4e^{-t/2} + C \end{aligned}$$

Multiply both sides by $e^{3t/2}$ to obtain the general solution for y.

$$y(t) = -\frac{2}{3}(3t+2) - 4e^t + Ce^{3t/2}$$

Now apply the initial condition to determine C.

$$y(0) = -\frac{2}{3}(2) - 4 + C = y_0 \quad \rightarrow \quad C = y_0 + \frac{16}{3}$$

Therefore, the solution to the initial value problem is

$$y(t) = -\frac{2}{3}(3t+2) - 4e^t + \left(y_0 + \frac{16}{3}\right)e^{3t/2}.$$

If $y_0 < -16/3$, then it will diverge to $-\infty$ as $t \rightarrow \infty$, and if $y_0 > -16/3$, then it will diverge to ∞ as $t \rightarrow \infty$. At the critical value of $y_0 = -16/3$, the solution reduces to

$$y(t) = -\frac{2}{3}(3t+2) - 4e^t,$$

which diverges to $-\infty$ as $t \rightarrow \infty$.