Problem 38

Variation of Parameters. Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \tag{i}$$

(a) If $g(t) = 0$ for all t, show that the solution is

$$y = A\exp\left[-\int p(t)\,dt\right], \tag{ii}$$

where A is a constant.

(b) If $g(t)$ is not everywhere zero, assume that the solution of Eq. (i) is of the form

$$y = A(t)\exp\left[-\int p(t)\,dt\right], \tag{iii}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t)\exp\left[\int p(t)\,dt\right]. \tag{iv}$$

(c) Find $A(t)$ from Eq. (iv). Then substitute for $A(t)$ in Eq. (iii) and determine y. Verify that the solution obtained in this manner agrees with that of Eq. (33) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.6 in connection with second order linear equations.

Solution

Part (a)

Suppose that $g(t) = 0$. Then equation (i) reduces to

$$y' + p(t)y = 0.$$

Divide both sides by y

$$\frac{y'}{y} + p(t) = 0$$

and bring $p(t)$ to the right side.

$$\frac{y'}{y} = -p(t)$$

The left side can be written as $d/dt(\ln|y|)$ by the chain rule. The absolute value sign is included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}(\ln|y|) = -p(t)$$

Integrate both sides with respect to t.

$$\ln|y| = \int^t [-p(s)]\,ds + C$$

Exponentiate both sides.

$$\begin{aligned}|y| &= \exp\left[-\int^t p(s)\,ds + C\right] \\ &= e^C \exp\left[-\int^t p(s)\,ds\right]\end{aligned}$$

Introduce ± on the right side to remove the absolute value sign.

$$y(t) = \pm e^C \exp\left[-\int^t p(s)\,ds\right]$$

Therefore, using a new constant A for $\pm e^C$,

$$y(t) = A\exp\left[-\int^t p(s)\,ds\right].$$

Part (b)

To solve the ODE for the case that $g(t)$ is nonzero, allow the parameter A to vary.

$$y(t) = A(t)\exp\left[-\int^t p(s)\,ds\right]$$

Substitute this formula into equation (i) to obtain an ODE for $A(t)$.

$$y' + p(t)y = g(t) \quad \rightarrow \quad \left\{A(t)\exp\left[-\int^t p(s)\,ds\right]\right\}' + p(t)\left\{A(t)\exp\left[-\int^t p(s)\,ds\right]\right\} = g(t)$$

Use the product rule to expand the left side.

$$A'(t)\exp\left[-\int^t p(s)\,ds\right] + A(t)\exp\left[-\int^t p(s)\,ds\right]\frac{d}{dt}\left[-\int^t p(s)\,ds\right] + p(t)A(t)\exp\left[-\int^t p(s)\,ds\right] = g(t)$$

$$A'(t)\exp\left[-\int^t p(s)\,ds\right] + \cancel{A(t)\exp\left[-\int^t p(s)\,ds\right][-p(t)]} + \cancel{p(t)A(t)\exp\left[-\int^t p(s)\,ds\right]} = g(t)$$

Multiply both sides by $\exp\left[\int^t p(s)\,ds\right]$ to solve for $A'(t)$.

$$A'(t) = g(t)\exp\left[\int^t p(s)\,ds\right]$$

Part (c)

Integrate both sides with respect to t to get $A(t)$.

$$A(t) = \int^t g(r)\exp\left[\int^r p(s)\,ds\right]dr + C_1$$

Now substitute this result into equation (iii).

$$\begin{aligned} y(t) &= A(t) \exp\left[-\int^t p(s)\,ds\right] \\ &= \frac{1}{\exp\left[\int^t p(s)\,ds\right]} A(t) \\ &= \frac{1}{\exp\left[\int^t p(s)\,ds\right]} \left\{\int^t g(r) \exp\left[\int^r p(s)\,ds\right] dr + C_1\right\} \end{aligned}$$

This is essentially equation (33) in the text,

$$y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^t \mu(s) g(s)\,ds + c\right], \tag{33}$$

considering that

$$\mu(t) = \exp\left[\int^t p(s)\,ds\right].$$