Problem 39

In each of Problems 39 through 42, use the method of Problem 38 to solve the given differential equation.

$$y' - 2y = t^2 e^{2t}$$

Solution

The method of variation of parameters will be used here. Start by solving the associated homogeneous equation.

$$Y' - 2Y = 0$$

Divide both sides by Y

$$\frac{Y'}{Y} - 2 = 0$$

and bring 2 to the right side.

$$\frac{Y'}{Y} = 2$$

The left side can be written as $d/dt(\ln|Y|)$ by the chain rule. The absolute value sign has been included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}(\ln|Y|) = 2$$

Integrate both sides with respect to t.

$$\ln|Y| = 2t + C$$

Exponentiate both sides.

$$\begin{aligned} |Y| &= e^{2t+C} \\ &= e^C e^{2t} \end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$Y(t) = \pm e^C e^{2t}$$

Therefore, using a new constant A for $\pm e^C$,

$$Y(t) = Ae^{2t}.$$

The solution for y is obtained by allowing the parameter A to vary.

$$y(t) = A(t)e^{2t} \tag{1}$$

Substitute this formula for y into the original ODE to obtain an equation for $A(t)$.

$$y' - 2y = t^2 e^{2t} \quad \to \quad [A(t)e^{2t}]' - 2[A(t)e^{2t}] = t^2 e^{2t}$$

Use the product rule to simplify the left side.

$$A'(t)e^{2t} + \cancel{A(t)(2e^{2t})} - \cancel{2A(t)e^{2t}} = t^2 e^{2t}$$

Divide both sides by e^{2t}.

$$A'(t) = t^2$$

Integrate both sides with respect to t.

$$A(t) = \frac{t^3}{3} + C_1$$

Substitute this result into equation (1) to obtain the general solution for y.

$$y(t) = \left(\frac{t^3}{3} + C_1\right) e^{2t}$$