Problem 19

In each of Problems 13 through 20, find the solution of the given initial value problem.

$$t^3y' + 4t^2y = e^{-t}, \qquad y(-1) = 0, \quad t < 0$$

Solution

Start by dividing both sides by t^3 to make the coefficient of y' 1.

$$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$$

This is a first-order linear inhomogeneous equation, so it can be solved with an integrating factor.

$$I(t) = \exp\left(\int^t \frac{4}{s}\,ds\right) = e^{4\ln t} = e^{\ln t^4} = t^4$$

Multiply both sides of the ODE by $I(t)$.

$$t^4y' + 4t^3y = te^{-t}$$

The left side can be written as $d/dt(Iy)$ by the product rule.

$$\frac{d}{dt}(t^4y) = te^{-t}$$

Integrate both sides with respect to t.

$$\begin{aligned} t^4y &= \int^t se^{-s}\,ds + C \\ &= -e^{-t}(1+t) + C \end{aligned}$$

Divide both sides by t^4 to solve for $y(t)$.

$$y(t) = -\frac{e^{-t}}{t^4}(1+t) + C$$

Apply the initial condition $y(-1) = 0$ here to determine C.

$$y(-1) = -e^1(0) + C = 0 \quad \rightarrow \quad C = 0$$

Therefore,

$$y(t) = -\frac{e^{-t}}{t^4}(1+t).$$