

## Problem 24

In each of Problems 24 through 26:

- Draw a direction field for the given differential equation. How do solutions appear to behave as  $t \rightarrow 0$ ? Does the behavior depend on the choice of the initial value  $a$ ? Let  $a_0$  be the value of  $a$  for which the transition from one type of behavior to another occurs. Estimate the value of  $a_0$ .
- Solve the initial value problem and find the critical value  $a_0$  exactly.
- Describe the behavior of the solution corresponding to the initial value  $a_0$ .

$$ty' + (t + 1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0$$

### Solution

#### Part (a)

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dt, dy \rangle = \left\langle 1, \frac{dy}{dt} \right\rangle dt = \left\langle 1, \frac{2te^{-t} - (t+1)y}{t} \right\rangle dt$$

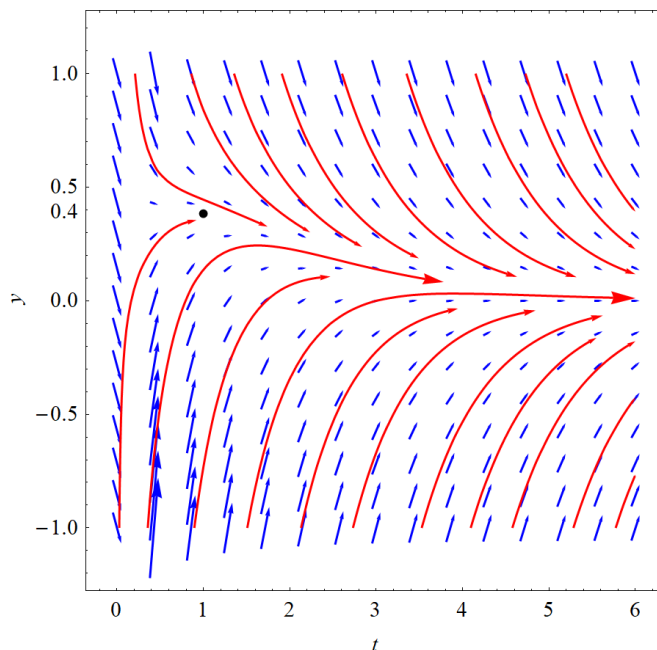


Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. The nature of the solutions appears to change for an initial condition of  $y(0) = a_0 \approx 0.4$ . Above this value,  $y \rightarrow \infty$  as  $t \rightarrow 0$ , and below this value,  $y \rightarrow -\infty$  as  $t \rightarrow 0$ .

**Part (b)**

$$ty' + (t + 1)y = 2te^{-t}$$

Divide both sides by  $t$  so that the coefficient of  $y'$  is 1.

$$y' + \left(1 + \frac{1}{t}\right)y = 2e^{-t}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor  $I$ .

$$I = \exp \left[ \int^t \left(1 + \frac{1}{s}\right) ds \right] = e^{t+\ln t} = te^t$$

Proceed with the multiplication.

$$te^t y' + (t + 1)e^t y = 2t$$

The left side can be written as  $d/dt(Iy)$  using the product rule.

$$\frac{d}{dt}(te^t y) = 2t$$

Integrate both sides with respect to  $t$ .

$$\begin{aligned} te^t y &= \int^t 2s ds + C \\ &= t^2 + C \end{aligned}$$

Divide both sides by  $te^t$  to obtain the general solution for  $y$ .

$$y(t) = e^{-t} \left( t + \frac{C}{t} \right)$$

Apply the initial condition  $y(1) = a$  now to determine  $C$ .

$$y(1) = e^{-1}(1 + C) = a \quad \rightarrow \quad C = ae - 1$$

Therefore, the solution to the initial value problem is

$$y(t) = e^{-t} \left( t + \frac{ae - 1}{t} \right).$$

If  $a$  is less than  $1/e$ , then  $y$  diverges to  $-\infty$  as  $t \rightarrow 0$ , and if  $a$  is greater than  $1/e$ , then  $y$  diverges to  $\infty$  as  $t \rightarrow 0$ . Therefore,

$$a_0 = \frac{1}{e} \approx 0.37.$$

**Part (c)**

If  $a = 1/e$ , then the solution to the initial value problem reduces to

$$y(t) = te^{-t},$$

which tends to zero as  $t \rightarrow \infty$ . It is plotted below as a function of  $t$ .

