Problem 33

Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

Solution

$$y' + ay = be^{-\lambda t}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t a\,ds\right) = e^{at}$$

Proceed with the multiplication.

$$e^{at}y' + ae^{at}y = be^{at}e^{-\lambda t}$$

The left side can be written as $d/dt(Iy)$ using the product rule.

$$\frac{d}{dt}(e^{at}y) = be^{(a-\lambda)t}$$

<u>**Case I:** $a = \lambda$</u>

$$\frac{d}{dt}(e^{at}y) = b$$

Integrate both sides with respect to t.

$$e^{at}y = bt + C$$

Divide both sides by e^{at} to obtain the general solution for y.

$$y(t) = \frac{bt}{e^{at}} + \frac{C}{e^{at}}$$

Take the limit of both sides as $t \to \infty$.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{bt}{e^{at}} + \underbrace{\lim_{t\to\infty} \frac{C}{e^{at}}}_{=\,0}$$

The remaining term is an indeterminate form (∞/∞), so l'Hôpital's rule can be applied to evaluate the limit.

$$\lim_{t\to\infty} y(t) \overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \lim_{t\to\infty} \frac{\frac{d}{dt}bt}{\frac{d}{dt}e^{at}} = \lim_{t\to\infty} \frac{b}{ae^{at}} = 0$$

We conclude that, in the case $a = \lambda$, all solutions of this ODE tend to zero as $t \to \infty$.

Case II: $a \neq \lambda$

$$\frac{d}{dt}(e^{at}y) = be^{(a-\lambda)t}$$

Integrate both sides with respect to t.

$$\begin{aligned} e^{at}y &= \int^t be^{(a-\lambda)s}\,ds + C_1 \\ &= \frac{b}{a-\lambda}e^{(a-\lambda)t} + C_1 \\ &= \frac{b}{a-\lambda}\frac{e^{at}}{e^{\lambda t}} + C_1 \end{aligned}$$

Divide both sides by e^{at} to obtain the general solution for y.

$$y(t) = \frac{b}{a-\lambda}\frac{1}{e^{\lambda t}} + \frac{C_1}{e^{at}}$$

Take the limit of both sides as $t \to \infty$.

$$\lim_{t\to\infty} y(t) = \underbrace{\lim_{t\to\infty} \frac{b}{a-\lambda}\frac{1}{e^{\lambda t}}}_{=\,0} + \underbrace{\lim_{t\to\infty} \frac{C_1}{e^{at}}}_{=\,0}$$

We conclude that, in the case $a \neq \lambda$, all solutions of this ODE tend to zero as $t \to \infty$. Therefore, every solution of $y' + ay = be^{-\lambda t}$ tends to zero as $t \to \infty$.