Problem 35

In each of Problems 34 through 37, construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.

Solution

The rate of change of y will become -1 as t gets big enough, so we choose

$$\begin{aligned} y' + y &= (-1) + (3 - t) \\ &= 2 - t. \end{aligned}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^t 1\,ds\right) = e^t$$

Proceed with the multiplication.

$$e^t y' + e^t y = e^t(2 - t)$$

The left side can be written as $d/dt(Iy)$ using the product rule.

$$\frac{d}{dt}(e^t y) = e^t(2 - t)$$

Integrate both sides with respect to t.

$$\begin{aligned} e^t y &= \int^t e^s(2 - s)\,ds + C \\ &= \int^t 2e^s\,ds - \int^t se^s\,ds + C \\ &= 2e^t - \int^t se^s\,ds + C \qquad (1) \end{aligned}$$

Use integration by parts to evaluate the remaining integral.

$$\begin{aligned} \int^t se^s\,ds &= \int^t s\frac{d}{ds}(e^s)\,ds \\ &= t(e^t) - \int^t (1)e^s\,ds \\ &= te^t - e^t \\ &= e^t(t - 1) \end{aligned}$$

Substitute this result into equation (1).

$$\begin{aligned} e^t y &= 2e^t - [e^t(t - 1)] + C \\ &= 2e^t - te^t + e^t + C \\ &= e^t(3 - t) + C \end{aligned}$$

Divide both sides by e^t to obtain the general solution for y.

$$y(t) = 3 - t + \frac{C}{e^t}$$

Take the limit of both sides as $t \to \infty$.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}(3 - t) + \underbrace{\lim_{t\to\infty} \frac{C}{e^t}}_{=\,0}$$

Therefore, all solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.