# Problem 36

In each of Problems 34 through 37, construct a first order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.

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### Solution

The rate of change of $y$ will become 2 as $t$ gets big enough, so we choose

$$\begin{aligned} y' + y &= (2) + (2t - 5) \\ &= 2t - 3. \end{aligned}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor $I$.

$$I = \exp\left(\int^t 1\,ds\right) = e^t$$

Proceed with the multiplication.

$$e^t y' + e^t y = e^t(2t - 3)$$

The left side can be written as $d/dt(Iy)$ using the product rule.

$$\frac{d}{dt}(e^t y) = e^t(2t - 3)$$

Integrate both sides with respect to $t$.

$$\begin{aligned} e^t y &= \int^t e^s(2s - 3)\,ds + C \\ &= \int^t 2se^s\,ds - \int^t 3e^s\,ds + C \\ &= 2\int^t se^s\,ds - 3e^t + C \end{aligned} \tag{1}$$

Use integration by parts to evaluate the remaining integral.

$$\begin{aligned} \int^t se^s\,ds &= \int^t s\frac{d}{ds}(e^s)\,ds \\ &= t(e^t) - \int^t (1)e^s\,ds \\ &= te^t - e^t \\ &= e^t(t - 1) \end{aligned}$$

Substitute this result into equation (1).

$$\begin{aligned} e^t y &= 2e^t(t - 1) - 3e^t + C \\ &= 2te^t - 5e^t + C \\ &= e^t(2t - 5) + C \end{aligned}$$

Divide both sides by $e^t$ to obtain the general solution for $y$.

$$y(t) = 2t - 5 + \frac{C}{e^t}$$

Take the limit of both sides as $t \to \infty$.

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (2t - 5) + \underbrace{\lim_{t\to\infty} \frac{C}{e^t}}_{=\,0}$$

Therefore, all solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.