Problem 41

In each of Problems 39 through 42, use the method of Problem 38 to solve the given differential equation.

$$ty' + 2y = \sin t, \qquad t > 0$$

Solution

The method of variation of parameters will be used here. Start by solving the associated homogeneous equation.

$$tY' + 2Y = 0$$

Divide both sides by tY

$$\frac{Y'}{Y} + \frac{2}{t} = 0$$

and bring $(2/t)$ to the right side.

$$\frac{Y'}{Y} = -\frac{2}{t}$$

The left side can be written as $d/dt(\ln|Y|)$ by the chain rule. The absolute value sign has been included because the argument of the logarithm cannot be negative.

$$\frac{d}{dt}(\ln|Y|) = -\frac{2}{t}$$

Integrate both sides with respect to t.

$$\ln|Y| = -2\ln t + C$$

Exponentiate both sides.

$$\begin{aligned}|Y| &= e^{-2\ln t + C} \\ &= e^C e^{\ln t^{-2}} \\ &= e^C t^{-2}\end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$Y(t) = \pm e^C t^{-2}$$

Therefore, using a new constant A for $\pm e^C$,

$$Y(t) = At^{-2}.$$

The solution for y is obtained by allowing the parameter A to vary.

$$y(t) = A(t)t^{-2} \tag{1}$$

Substitute this formula for y into the original ODE to obtain an equation for $A(t)$.

$$ty' + 2y = \sin t \quad \rightarrow \quad t[A(t)t^{-2}]' + 2[A(t)t^{-2}] = \sin t$$

Use the product rule to simplify the left side.

$$t[A'(t)t^{-2} + A(t)(-2t^{-3})] + 2A(t)t^{-2} = \sin t$$

$$A'(t)t^{-1} - \cancel{2A(t)t^{-2}} + \cancel{2A(t)t^{-2}} = \sin t$$

Multiply both sides by t.

$$A'(t) = t\sin t$$

Integrate both sides with respect to t.

$$\begin{aligned}
A(t) &= \int^t s\sin s\,ds + C_1 \\
&= \int^t s\frac{d}{ds}(-\cos s)\,ds + C_1 \\
&= t(-\cos t) - \int^t (1)(-\cos s)\,ds + C_1 \\
&= -t\cos t + \int^t \cos s\,ds + C_1 \\
&= -t\cos t + \sin t + C_1
\end{aligned}$$

Substitute this result into equation (1) to obtain the general solution for y.

$$y(t) = (-t\cos t + \sin t + C_1)t^{-2}$$