Problem 11

In each of Problems 9 through 20:

(a) Find the solution of the given initial value problem in explicit form.

(b) Plot the graph of the solution.

(c) Determine (at least approximately) the interval in which the solution is defined.

$$x\,dx + ye^{-x}\,dy = 0, \qquad y(0) = 1$$

Solution

Part (a)

Solve the ODE for dy/dx.

$$x\,dx + ye^{-x}\,dy = 0 \quad \rightarrow \quad ye^{-x}\,dy = -x\,dx$$

Actually, we see that the variables, x and y, become separated by multiplying both sides by e^x.

$$y\,dy = -xe^x\,dx$$

Integrate both sides, using integration by parts for the integral in dx.

$$\int y\,dy = \int (-xe^x)\,dx$$

$$\begin{aligned}
\frac{y^2}{2} &= -\int xe^x\,dx \\
&= -\int x\frac{d}{dx}(e^x)\,dx \\
&= -\left[x(e^x) - \int (1)(e^x)\,dx\right] \\
&= -\left(xe^x - \int e^x\,dx\right) \\
&= -(xe^x - e^x) + C \\
&= -xe^x + e^x + C \\
&= e^x(1-x) + C
\end{aligned}$$

Now apply the initial condition $y(0) = 1$ to determine C.

$$\frac{1^2}{2} = e^0(1-0) + C \quad \rightarrow \quad \frac{1}{2} = 1 + C \quad \rightarrow \quad C = -\frac{1}{2}$$

As a result,

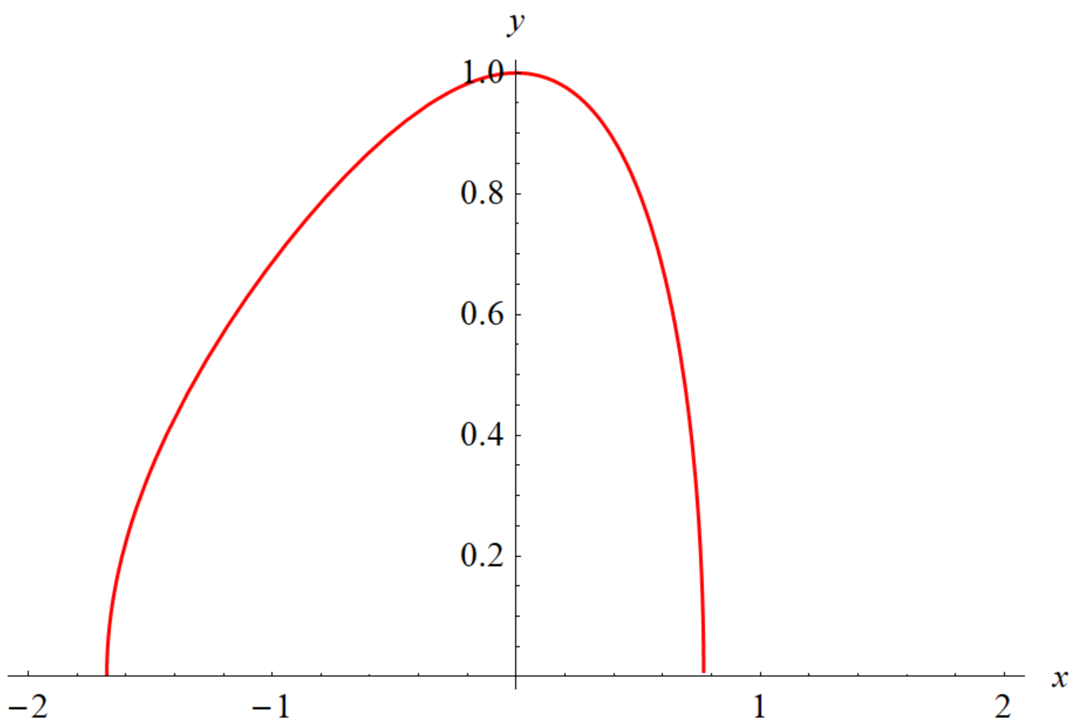
$$\frac{y^2}{2} = e^x(1-x) - \frac{1}{2}$$
$$y^2 = 2e^x(1-x) - 1$$
$$y(x) = \pm\sqrt{2e^x(1-x)-1}.$$

We choose the plus sign so that the initial condition is satisfied. Therefore,

$$y(x) = \sqrt{2e^x(1-x)-1}.$$

Part (b)

Below is a plot of $y(x)$ versus x.



Part (c)

The solution we found is only valid along the curve that passes through $x = 0$ (the y-axis) because $x = 0$ is where the initial condition is given. Judging from the graph, it is valid over approximately $-1.67 \lesssim x \lesssim 0.77$, the domain of y.