Problem 12

In each of Problems 9 through 20:

(a) Find the solution of the given initial value problem in explicit form.

(b) Plot the graph of the solution.

(c) Determine (at least approximately) the interval in which the solution is defined.

$$dr/d\theta = r^2/\theta, \qquad r(1) = 2$$

Solution

This ODE is separable because it is of the form $r' = f(r)g(\theta)$, so it can be solved by separating variables. Bring the terms with r to the left and bring the terms with θ to the right.

$$\frac{dr}{r^2} = \frac{d\theta}{\theta}$$

Integrate both sides.

$$\int \frac{dr}{r^2} = \int \frac{d\theta}{\theta}$$

$$-\frac{1}{r} = \ln|\theta| + C$$

Now apply the initial condition $r(1) = 2$ to determine C.

$$-\frac{1}{2} = \ln|1| + C \quad \rightarrow \quad -\frac{1}{2} = C$$

The previous equation becomes

$$-\frac{1}{r} = \ln|\theta| - \frac{1}{2}$$

$$\frac{1}{r} = \frac{1}{2} - \ln|\theta|.$$

Therefore,

$$r(\theta) = \frac{1}{\frac{1}{2} - \ln|\theta|}.$$

Since r must be positive, the solution is defined where

$$\frac{1}{2} - \ln|\theta| > 0$$

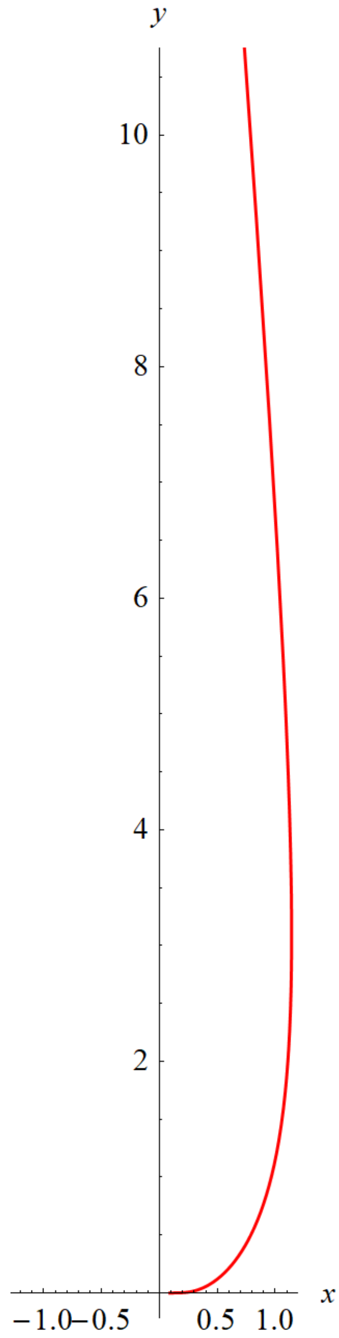
$$\ln|\theta| < \frac{1}{2}$$

$$|\theta| < e^{1/2}$$

$$0 < \theta < e^{1/2}.$$

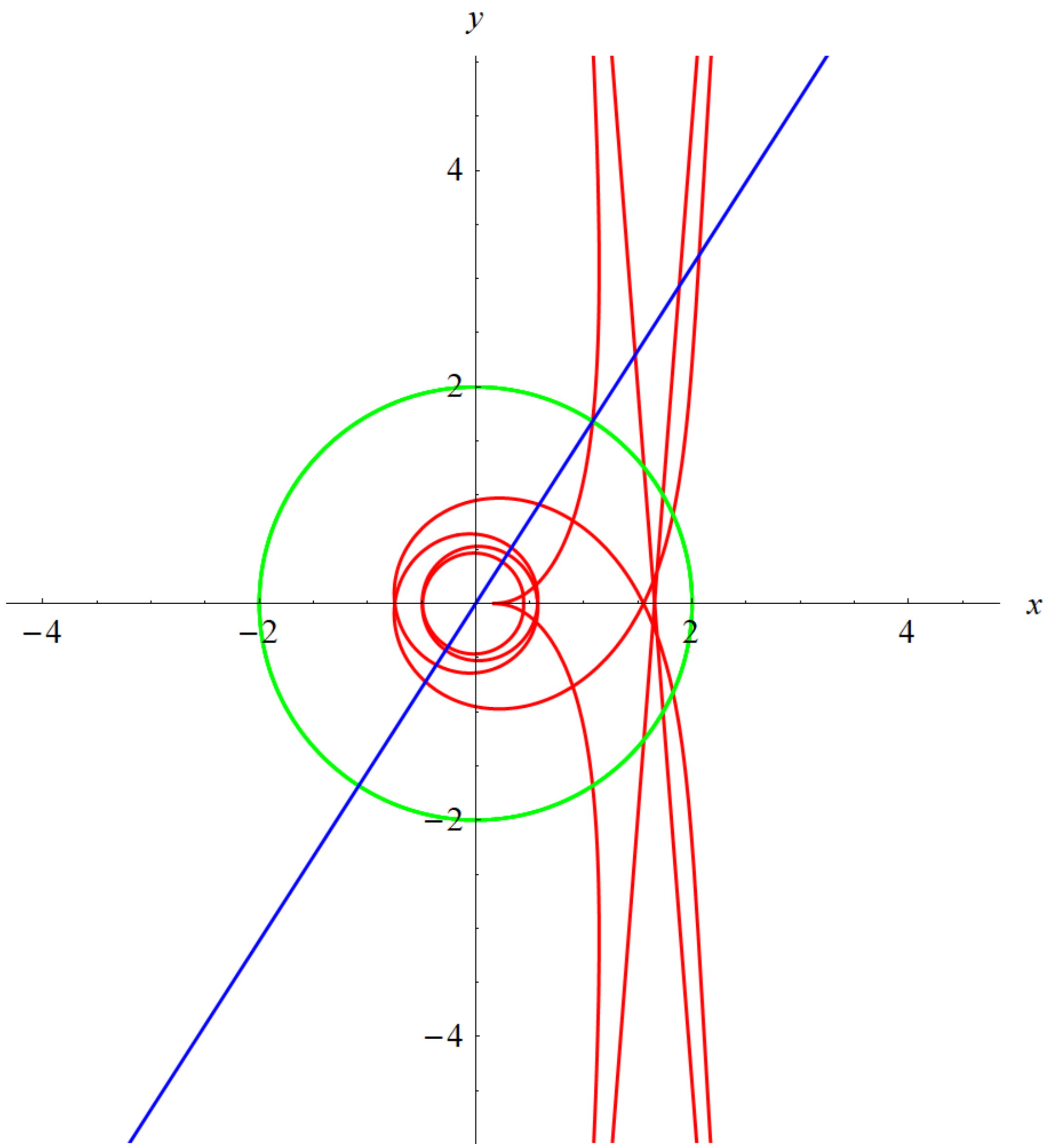
Part (b)

Below is a plot of $r(\theta)$ versus θ for $0 < \theta < e^{1/2}$.



Part (c)

The solution we found is only valid along the curve that passes through $r = 2$ and $\theta = 1$. Below in red is a plot of $r(\theta)$ versus θ for $-5\pi < \theta < 5\pi$.



In green is the circle $r = 2$, and in blue is the line at an angle of 1 radian above the x-axis. The red curve that the circle and line intersect on is the one that is plotted in part (b).