Problem 38

The method outlined in Problem 30 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 31 through 38:

(a) Show that the given equation is homogeneous.

(b) Solve the differential equation.

(c) Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$$

Solution

Part (a)

Multiply the numerator and denominator by $1/x^2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{3y^2 - x^2}{2xy} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\ &= \frac{3\frac{y^2}{x^2} - 1}{2\frac{y}{x}}\end{aligned}$$

Because dy/dx can be written in terms of y/x, the ODE is homogeneous.

Part (b)

Make the change of variables,

$$\begin{aligned} v &= \frac{y}{x} & &\rightarrow & y &= xv \\ \frac{dv}{dx} &= \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} & &\rightarrow & \frac{dy}{dx} &= x\frac{dv}{dx} + \frac{y}{x} = x\frac{dv}{dx} + v. \end{aligned}$$

As a result, the ODE becomes

$$\frac{dy}{dx} = \frac{3\frac{y^2}{x^2} - 1}{2\frac{y}{x}} \quad \rightarrow \quad x\frac{dv}{dx} + v = \frac{3v^2 - 1}{2v}.$$

Bring v to the right side and simplify.

$$\begin{aligned} x\frac{dv}{dx} &= \frac{3v^2 - 1}{2v} - v \\ &= \frac{3v^2 - 1 - v(2v)}{2v} \\ &= \frac{v^2 - 1}{2v} \end{aligned}$$

Separate variables by bringing the terms with v to the left side and bringing the terms with x to the right side.

$$\frac{2v}{v^2-1}\,dv = \frac{dx}{x}$$

Integrate both sides.

$$\int \frac{2v}{v^2-1}\,dv = \int \frac{dx}{x}$$

Let $w = v^2 - 1$ so that $dw = 2v\,dv$.

$$\int \frac{dw}{w} = \int \frac{dx}{x}$$

$$\ln|w| = \ln|x| + C$$

$$\ln|v^2-1| = \ln|x| + C$$

$$\ln|v^2-1| - \ln|x| = C$$

$$\ln\left|\frac{v^2-1}{x}\right| = C$$

Exponentiate both sides.

$$\left|\frac{v^2-1}{x}\right| = e^C$$

Introduce $\pm$ on the right to remove the absolute value sign.

$$\frac{v^2-1}{x} = \pm e^C$$

Use a new constant A for $\pm e^C$.

$$\frac{v^2-1}{x} = A$$

Solve for v.

$$v^2 = Ax + 1$$

$$v(x) = \pm\sqrt{Ax+1}$$

Replace v with y/x.

$$\frac{y}{x} = \pm\sqrt{Ax+1}$$

Therefore,

$$y(x) = \pm x\sqrt{Ax+1}.$$

Part (c)

The direction field is a two-dimensional vector field that shows what the direction of the solution is at every point in a region. Every solution to the differential equation is a curve drawn such that the direction field vectors are tangent to it at every point.

$$\langle dx, dy\rangle = \left\langle 1, \frac{dy}{dx}\right\rangle dx = \left\langle 1, \frac{3y^2-x^2}{2xy}\right\rangle dx$$

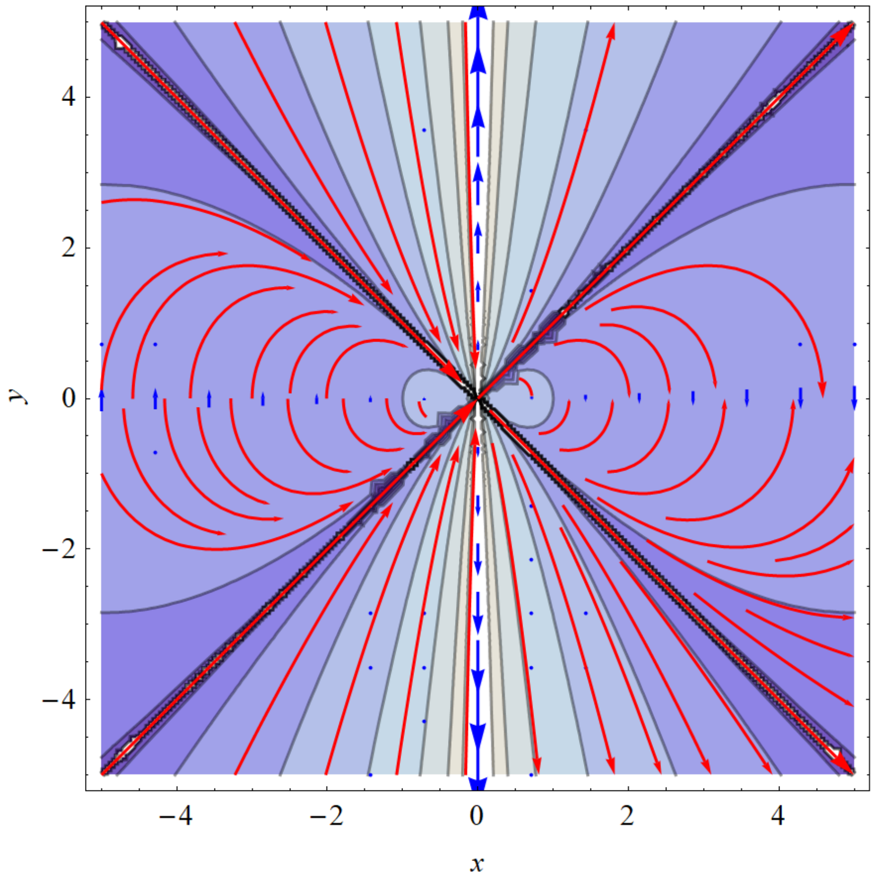

Figure 1: In blue are the direction field vectors and in red are possible solutions to the differential equation, depending what the initial condition is. This plot does have symmetry with respect to the origin.