Problem 14

In each of Problems 9 through 20:

(a) Find the solution of the given initial value problem in explicit form.

(b) Plot the graph of the solution.

(c) Determine (at least approximately) the interval in which the solution is defined.

$$y' = xy^3(1+x^2)^{-1/2}, \qquad y(0) = 1$$

Solution

Part (a)

This ODE is separable because it is of the form $y' = f(x)g(y)$, so it can be solved by separating variables.

$$\frac{dy}{dx} = \frac{xy^3}{(1+x^2)^{1/2}}$$

Bring the terms with y to the left and bring the terms with x to the right.

$$\frac{dy}{y^3} = \frac{x}{(1+x^2)^{1/2}}\,dx$$

Integrate both sides.

$$\int \frac{dy}{y^3} = \int \frac{x}{(1+x^2)^{1/2}}\,dx \tag{1}$$

Make the following substitution in the integral in dx.

$$u = 1 + x^2$$

$$du = 2x\,dx \quad \rightarrow \quad \frac{du}{2} = x\,dx$$

Equation (1) becomes

$$\begin{aligned} -\frac{1}{2y^2} &= \int \frac{1}{u^{1/2}}\left(\frac{du}{2}\right) \\ &= \frac{1}{2}\int u^{-1/2}\,du \\ &= u^{1/2} + C \\ &= (1+x^2)^{1/2} + C. \end{aligned}$$

Now apply the initial condition to determine C.

$$-\frac{1}{2(1)^2} = (1+0)^{1/2} + C \quad \rightarrow \quad C = -\frac{3}{2}$$

As a result,

$$\begin{aligned} -\frac{1}{2y^2} &= (1+x^2)^{1/2} - \frac{3}{2} \\ &= -\frac{1}{2}(3 - 2\sqrt{1+x^2}). \end{aligned}$$

Solve for y.

$$\frac{1}{y^2} = 3 - 2\sqrt{1+x^2}$$

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}}$$

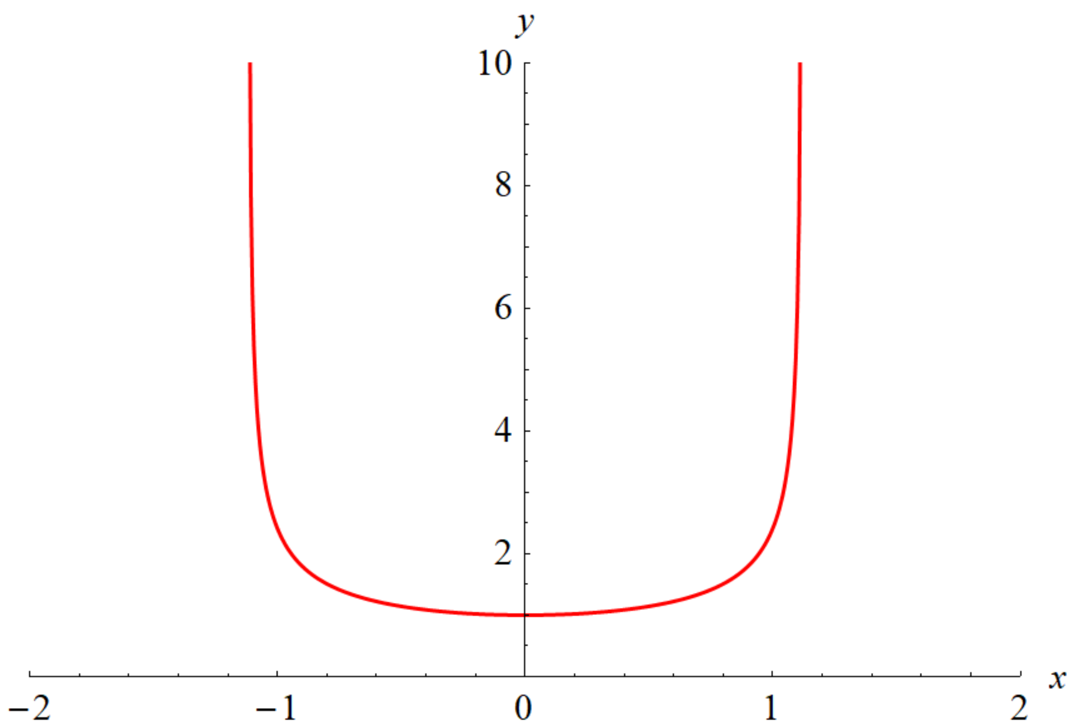
$$y(x) = \pm\sqrt{\frac{1}{3 - 2\sqrt{1+x^2}}}$$

We choose the plus sign so that the initial condition is satisfied. Therefore,

$$y(x) = \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}}.$$

Part (b)

Below is a plot of $y(x)$ versus x.



Part (c)

The solution we found is only valid along the curve that passes through $x = 0$ (the y-axis) because $x = 0$ is where the initial condition is given. It is valid over the domain of y.

$$3 - 2\sqrt{1+x^2} > 0$$

$$-2\sqrt{1+x^2} > -3$$

$$\sqrt{1+x^2} < \frac{3}{2}$$

$$1 + x^2 < \frac{9}{4}$$

$$x^2 < \frac{5}{4}$$

$$|x| < \frac{\sqrt{5}}{2} \approx 1.12$$

Therefore, the solution is valid over

$$-\frac{\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}.$$