Problem 22

Solve the initial value problem

$$y' = 3x^2/(3y^2 - 4), \qquad y(1) = 0$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

Solution

This ODE is separable because it is of the form $y' = f(x)g(y)$, so it can be solved by separating variables.

$$\frac{dy}{dx} = \frac{3x^2}{3y^2 - 4}$$

Bring the terms with y to the left and bring the terms with x to the right.

$$(3y^2 - 4)\,dy = 3x^2\,dx$$

Integrate both sides.

$$\int (3y^2 - 4)\,dy = \int 3x^2\,dx$$

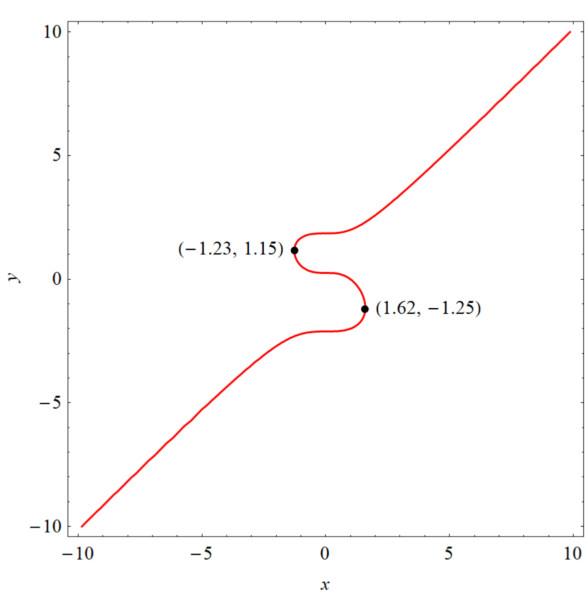
$$y^3 - 4y = x^3 + C$$

Use the initial condition now to determine C.

$$0 - 0 = 1^3 + C \quad \to \quad C = -1$$

Therefore, an implicit solution to the ODE is

$$y^3 - 4y = x^3 - 1.$$



Factor the denominator in the ODE.

$$\frac{dy}{dx} = \frac{3x^2}{(\sqrt{3}y + 2)(\sqrt{3}y - 2)}$$

The denominator blows up when $(\sqrt{3}y + 2)(\sqrt{3}y - 2) = 0$, that is,

$$y = -\frac{2}{\sqrt{3}} \quad \text{or} \quad y = \frac{2}{\sqrt{3}}.$$

Use the solution to the ODE to find the values of x that correspond to these values of y.

$$\begin{aligned} y &= -\frac{2}{\sqrt{3}}: \quad \frac{16}{3\sqrt{3}} = x^3 - 1 \quad \rightarrow \quad x = \sqrt[3]{1 + \frac{16}{3\sqrt{3}}} \approx 1.6 \\ y &= \frac{2}{\sqrt{3}}: \quad -\frac{16}{3\sqrt{3}} = x^3 - 1 \quad \rightarrow \quad x = \sqrt[3]{1 - \frac{16}{3\sqrt{3}}} \approx -1.3 \end{aligned}$$

The solution we found is valid on the part of the curve that passes through the point $(x = 1, y = 0)$ until dy/dx becomes unbounded.

$$\sqrt[3]{1 - \frac{16}{3\sqrt{3}}} < x < \sqrt[3]{1 + \frac{16}{3\sqrt{3}}}$$