Problem 23

Solve the initial value problem

$$y' = 2y^2 + xy^2, \qquad y(0) = 1$$

and determine where the solution attains its minimum value.

Solution

Factor y^2 from the right side.

$$\frac{dy}{dx} = y^2(2 + x)$$

This ODE is separable because it is of the form $y' = f(x)g(y)$, so it can be solved by separating variables. Bring the terms with y to the left and bring the terms with x to the right.

$$\frac{dy}{y^2} = (2 + x)\,dx$$

Integrate both sides.

$$\int \frac{dy}{y^2} = \int (2 + x)\,dx$$

$$-\frac{1}{y} = 2x + \frac{x^2}{2} + C$$

Apply the initial condition now to determine C.

$$-\frac{1}{1} = 0 + 0 + C \quad \rightarrow \quad C = -1$$

As a result,

$$-\frac{1}{y} = 2x + \frac{x^2}{2} - 1$$

$$\frac{1}{y} = 1 - 2x - \frac{x^2}{2}$$

$$y(x) = \frac{1}{1 - 2x - \frac{x^2}{2}}.$$

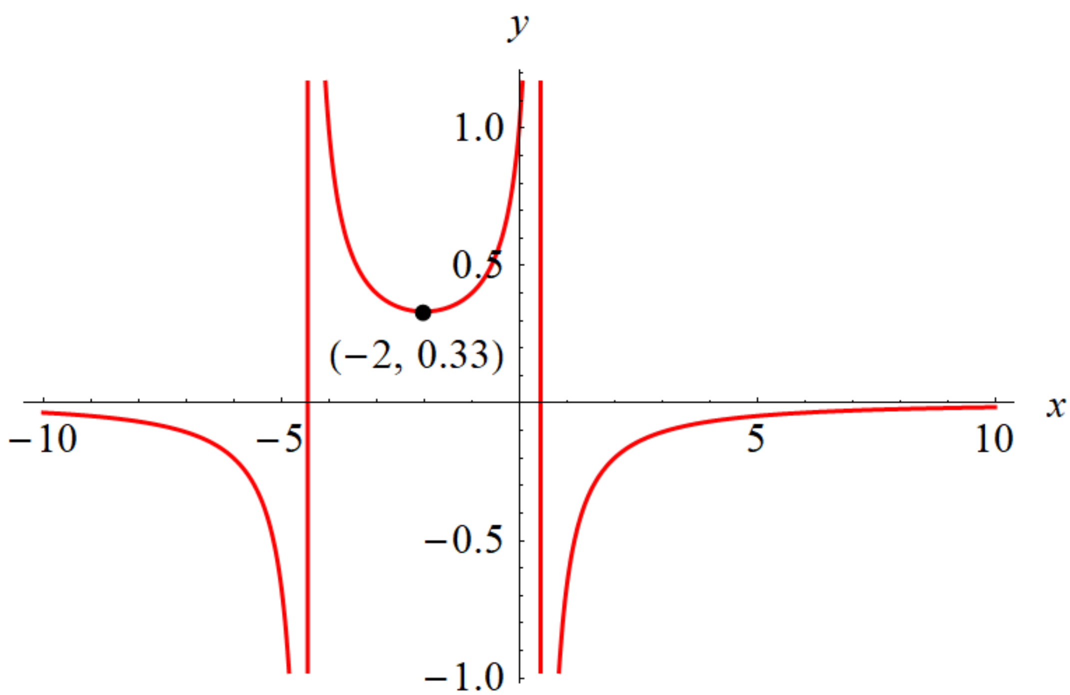
Therefore,

$$y(x) = \frac{2}{2 - 4x - x^2}.$$

Inspecting the ODE, we see that $dy/dx = 0$ when $x = -2$. Plugging this value into the solution, we get

$$y(-2) = \frac{2}{2 - 4(-2) - (-2)^2} = \frac{1}{3}.$$

Therefore, the solution attains its minimum value at $\left(-2, \frac{1}{3}\right)$.


y
1.0
0.5
(−2, 0.33)
x
−10
−5
5
10
−0.5
−1.0