Problem 25

Solve the initial value problem

$$y' = 2\cos 2x/(3+2y), \qquad y(0) = -1$$

and determine where the solution attains its maximum value.

Solution

This ODE is separable because it is of the form $y' = f(x)g(y)$, so it can be solved by separating variables.

$$\frac{dy}{dx} = \frac{2\cos 2x}{3+2y}$$

Bring the terms with y to the left and bring the terms with x to the right.

$$(3+2y)\,dy = 2\cos 2x\,dx$$

Integrate both sides.

$$\int (3+2y)\,dy = \int 2\cos 2x\,dx$$

$$3y + y^2 = \sin 2x + C$$

Apply the initial condition now to determine C.

$$3(-1) + (-1)^2 = 0 + C \quad \to \quad C = -2$$

As a result,

$$3y + y^2 = \sin 2x - 2$$

$$y^2 + 3y - (\sin 2x - 2) = 0$$

$$y(x) = \frac{-3 \pm \sqrt{9 + 4(\sin 2x - 2)}}{2}$$

We choose the plus sign so that the initial condition is satisfied. Therefore,

$$y(x) = \frac{-3 + \sqrt{1 + 4\sin 2x}}{2}.$$

Inspecting the solution, we see that the maximum occurs when $\sin 2x = 1$, or

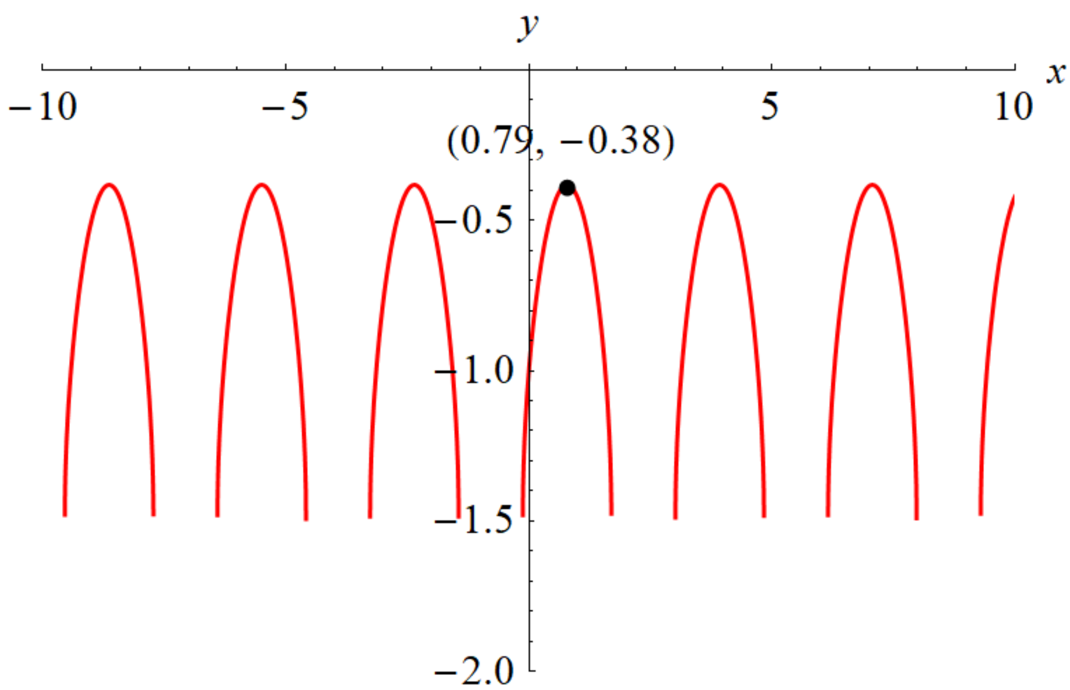
$$2x = \frac{\pi}{2} + 2n\pi,\ n = 0, \pm 1, \pm 2, \ldots$$

$$x = \frac{\pi}{4} + n\pi.$$

At this value of x, the solution is

$$y\left(\frac{\pi}{4} + n\pi\right) = \frac{-3 + \sqrt{5}}{2} \approx -0.38,$$

which means the maxima occur at approximately $\left(\frac{\pi}{4} + n\pi, -0.38\right)$ for $n = 0, \pm 1, \pm 2, \ldots$.



Since the solution is only valid on the part of the curve that passes through $x = 0$ (the y-axis), the maximum we're interested in is the $n = 0$ one: $(0.79, -0.38)$.