Problem 26

Solve the initial value problem

$$y' = 2(1+x)(1+y^2), \qquad y(0) = 0$$

and determine where the solution attains its minimum value.

Solution

This ODE is separable because it is of the form $y' = f(x)g(y)$, so it can be solved by separating variables.

$$\frac{dy}{dx} = 2(1+x)(1+y^2)$$

Bring the terms with y to the left and bring the terms with x to the right.

$$\frac{dy}{1+y^2} = 2(1+x)\,dx$$

Integrate both sides.

$$\int \frac{dy}{1+y^2} = \int 2(1+x)\,dx$$

$$\tan^{-1} y = 2x + x^2 + C$$

Apply the initial condition now to determine C.

$$0 = 0 + 0 + C \quad \rightarrow \quad C = 0$$

As a result,

$$\tan^{-1} y = 2x + x^2.$$

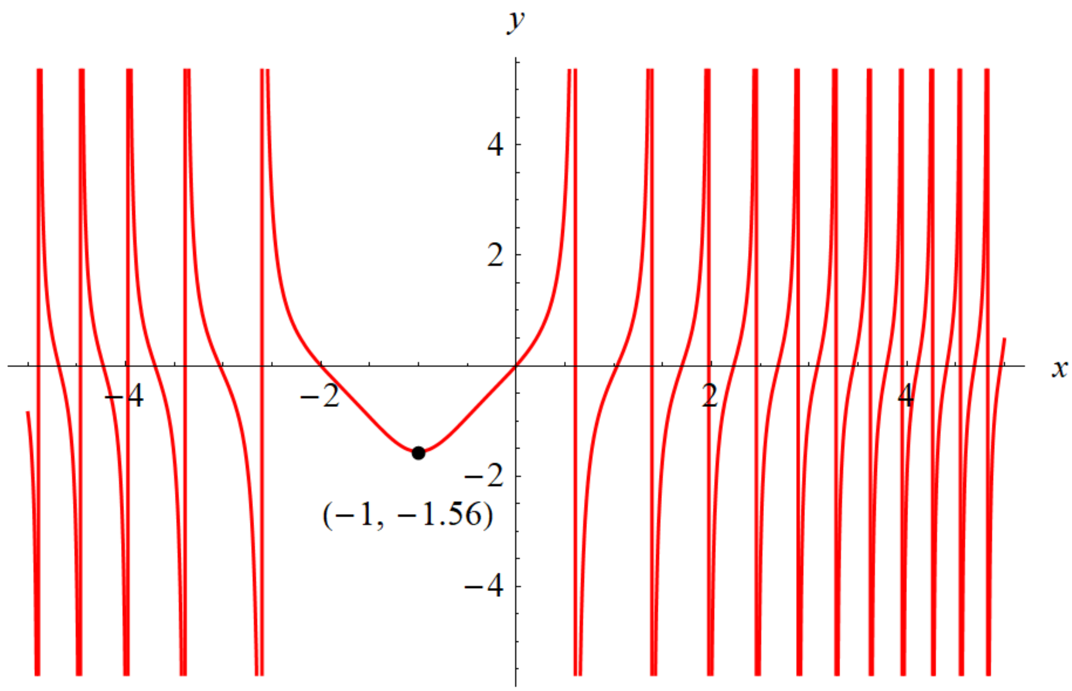
Therefore, taking the tangent of both sides,

$$y(x) = \tan(2x + x^2).$$

Inspecting the ODE, we see that $dy/dx = 0$ when $x = -1$. Substitute this value of x to find the corresponding value for y.

$$y(-1) = \tan(-1) \approx -1.56$$

Therefore, the minimum occurs at approximately $(-1, -1.56)$.



The solution we found is only valid on the part of the curve that passes through $x = 0$ (the y-axis). The minimum is marked.