Problem 27

Consider the initial value problem

$$y' = ty(4-y)/3, \qquad y(0) = y_0.$$

(a) Determine how the behavior of the solution as t increases depends on the initial value y_0.

(b) Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

Solution

Part (a)

This ODE is separable because it is of the form $y' = f(t)g(y)$, so it can be solved by separating variables.

$$\frac{dy}{dt} = \frac{ty(4-y)}{3}$$

Bring the terms with y to the left and bring the terms with t to the right.

$$\frac{dy}{y(4-y)} = \frac{t}{3}\,dt$$

Integrate both sides.

$$\int \frac{dy}{y(4-y)} = \int \frac{t}{3}\,dt$$

$$\int \frac{1}{4}\left(\frac{1}{y} + \frac{1}{4-y}\right) dy = \frac{t^2}{6} + C$$

$$\frac{1}{4}\int \left(\frac{1}{y} - \frac{1}{y-4}\right) dy = \frac{t^2}{6} + C$$

Multiply both sides by 4 and evaluate the integral.

$$\ln|y| - \ln|y-4| = \frac{2}{3}t^2 + 4C$$

$$\ln\left|\frac{y}{y-4}\right| = \frac{2}{3}t^2 + 4C$$

Exponentiate both sides.

$$\left|\frac{y}{y-4}\right| = \exp\left(\frac{2t^2}{3} + 4C\right)$$
$$= e^{4C}e^{2t^2/3}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$\frac{y}{y-4} = \pm e^{4C}e^{2t^2/3}$$

Use a new constant A for $\pm e^{4C}$.

$$\frac{y}{y-4} = Ae^{2t^2/3}$$

Apply the initial condition $y(0) = y_0$ now to determine A.

$$\frac{y_0}{y_0 - 4} = A$$

As a result,

$$\frac{y}{y-4} = \frac{y_0}{y_0 - 4} e^{2t^2/3}.$$

Multiply both sides by $y - 4$.

$$y = y\frac{y_0}{y_0 - 4} e^{2t^2/3} - 4\frac{y_0}{y_0 - 4} e^{2t^2/3}$$

$$y\left(1 - \frac{y_0}{y_0 - 4} e^{2t^2/3}\right) = -4\frac{y_0}{y_0 - 4} e^{2t^2/3}$$

Therefore, the solution is

$$\begin{aligned} y(t) &= \frac{-4\frac{y_0}{y_0-4} e^{2t^2/3}}{1 - \frac{y_0}{y_0-4} e^{2t^2/3}} \\ &= \frac{4y_0 e^{2t^2/3}}{4 - y_0 + y_0 e^{2t^2/3}} \\ &= \frac{4y_0 e^{2t^2/3}}{4 + y_0(e^{2t^2/3} - 1)}. \end{aligned}$$

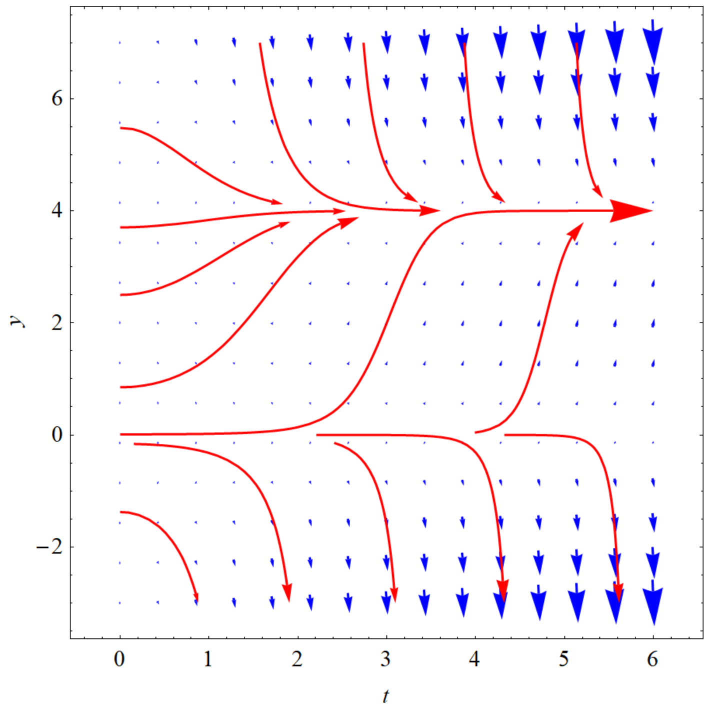

Figure 1: This figure shows the direction field for the ODE. In red are possible solutions to the ODE, depending what y_0 is. If $y_0 > 0$, then the solution converges to $y = 4$. Otherwise, the solution diverges to $-\infty$.

Part (b)

Set $y_0 = 0.5$ and $y = 3.98$ and solve the equation for t.

$$3.98 = \frac{4(0.5)e^{2t^2/3}}{4 + 0.5(e^{2t^2/3} - 1)}$$

$$3.98 = \frac{4e^{2t^2/3}}{8 + e^{2t^2/3} - 1}$$

Multiply both sides by $8 + e^{2t^2/3} - 1$.

$$31.84 + 3.98e^{2t^2/3} - 3.98 = 4e^{2t^2/3}$$

$$27.86 = 0.02e^{2t^2/3}$$

$$1393 = e^{2t^2/3}$$

$$\ln 1393 = \frac{2t^2}{3}$$

$$t^2 = \frac{3}{2}\ln 1393$$

Therefore,

$$t = \sqrt{\frac{3}{2}\ln 1393} \approx 3.30.$$