Problem 28

Consider the initial value problem

$$y' = ty(4-y)/(1+t), \qquad y(0) = y_0 > 0.$$

(a) Determine how the solution behaves as $t \to \infty$.

(b) If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.

(c) Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

Solution

Part (a)

This ODE is separable because it is of the form $y' = f(t)g(y)$, so it can be solved by separating variables.

$$\frac{dy}{dt} = \frac{ty(4-y)}{1+t}$$

Bring the terms with y to the left and bring the terms with t to the right.

$$\frac{dy}{y(4-y)} = \frac{t}{1+t}\,dt$$

Integrate both sides.

$$\int \frac{dy}{y(4-y)} = \int \frac{t}{1+t}\,dt$$

Make the substitution $u = 1+t$ in the integral on the right.

$$\int \frac{1}{4}\left(\frac{1}{y} + \frac{1}{4-y}\right)dy = \int \frac{u-1}{u}\,du$$

$$\frac{1}{4}\int \left(\frac{1}{y} - \frac{1}{y-4}\right)dy = u - \ln|u| + C$$

$$\frac{1}{4}(\ln|y| - \ln|y-4|) = 1 + t - \ln(1+t) + C$$

Use a new constant C_1 for $1 + C$.

$$\frac{1}{4}\ln\left|\frac{y}{y-4}\right| = t - \ln(1+t) + C_1$$

Multiply both sides by 4.

$$\begin{aligned}\ln\left|\frac{y}{y-4}\right| &= 4t - 4\ln(1+t) + 4C_1 \\ &= 4t + \ln(1+t)^{-4} + 4C_1\end{aligned}$$

Exponentiate both sides.

$$\begin{aligned}\left|\frac{y}{y-4}\right| &= e^{4t+\ln(1+t)^{-4}+4C_1} \\ &= e^{4t}(1+t)^{-4}e^{4C_1}\end{aligned}$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$\frac{y}{y-4} = \pm e^{4t}(1+t)^{-4}e^{4C_1}$$

Use a new constant A for $\pm e^{4C_1}$.

$$\frac{y}{y-4} = A\frac{e^{4t}}{(1+t)^4}$$

Apply the initial condition $y(0) = y_0$ here to determine A.

$$\frac{y_0}{y_0-4} = A$$

As a result,

$$\frac{y}{y-4} = \frac{y_0}{y_0-4}\frac{e^{4t}}{(1+t)^4}.$$

Multiply both sides by $y-4$.

$$y = y\frac{y_0}{y_0-4}\frac{e^{4t}}{(1+t)^4} - 4\frac{y_0}{y_0-4}\frac{e^{4t}}{(1+t)^4}$$

$$y\left[1 - \frac{y_0}{y_0-4}\frac{e^{4t}}{(1+t)^4}\right] = -4\frac{y_0}{y_0-4}\frac{e^{4t}}{(1+t)^4}$$

Therefore, the solution is

$$\begin{aligned} y(t) &= \frac{-4\frac{y_0}{y_0-4}\frac{e^{4t}}{(1+t)^4}}{1-\frac{y_0}{y_0-4}\frac{e^{4t}}{(1+t)^4}} \\ &= \frac{4y_0e^{4t}}{(4-y_0)(t+1)^4 + y_0e^{4t}} \\ &= \frac{4y_0}{(4-y_0)(t+1)^4e^{-4t} + y_0}. \end{aligned}$$

In the limit as $t \to \infty$, the first term in the denominator vanishes because of the decaying exponential function.

$$\lim_{t\to\infty} y(t) = \frac{4y_0}{0+y_0} = 4$$

Part (b)

Set $y_0 = 2$ and $y = 3.99$ and solve the equation for t numerically.

$$3.99 = \frac{4(2)}{2(t+1)^4e^{-4t} + 2}$$

$$t \approx 2.84$$

Part (c)

Set $t = 2$ in the solution and solve the following inequality for y_0.

$$3.99 < y(2) < 4.01$$

$$3.99 < \frac{4y_0}{(4 - y_0)(3)^4 e^{-4(2)} + y_0} < 4.01$$

Split up the inequality into two.

$$\frac{4y_0}{81(4 - y_0)e^{-8} + y_0} > 3.99 \qquad \text{and} \qquad \frac{4y_0}{81(4 - y_0)e^{-8} + y_0} < 4.01$$

$$\frac{4y_0}{81(4 - y_0)e^{-8} + y_0} - 3.99 > 0 \qquad \text{and} \qquad \frac{4y_0}{81(4 - y_0)e^{-8} + y_0} - 4.01 < 0$$

$$\frac{0.01y_0 - 323.19(4 - y_0)e^{-8}}{81(4 - y_0)e^{-8} + y_0} > 0 \qquad \text{and} \qquad \frac{-0.01y_0 - 324.81(4 - y_0)e^{-8}}{81(4 - y_0)e^{-8} + y_0} < 0$$

Determine the critical points by finding where the numerators are zero.

$$\text{Critical Point: } y_0 \approx 3.66 \qquad \text{and} \qquad \text{Critical Point: } y_0 \approx 4.40$$

Testing values of y_0 above and below the critical points, we find where each inequality is true.

$$y_0 \gtrsim 3.66 \qquad \text{and} \qquad y_0 \lesssim 4.40$$

Therefore,

$$3.66 \lesssim y_0 \lesssim 4.40.$$